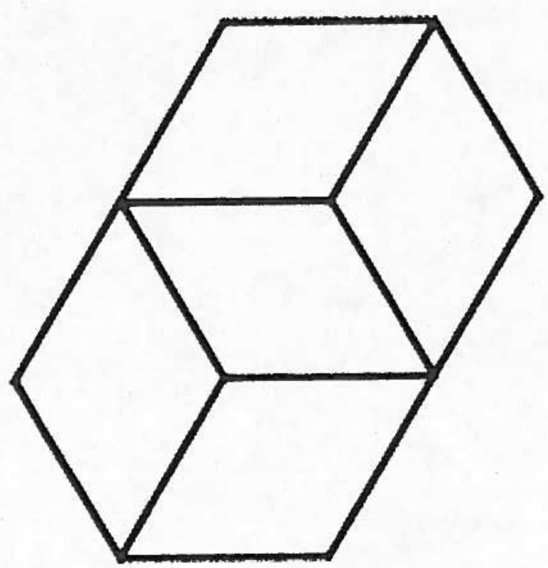
Quick Draw

Developing Spatial Sense in Mathematics

Grayson H. Wheatley
Florida State University



Mathematics Learning
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Introduction

Just as it is useful to have a mental map of the streets to plan the best route in traveling across town, it is useful to have mental images of mathematical patterns and relationships. If a student has constructed a network of imaged-based mathematical meanings, then solutions to problems can be more easily devised. If a person only knows one way and the task is not straight forward, solving the problem may not be possible. In doing mathematics, it is beneficial to recognize that there is more than one way to solve a problem or complete a routine task. It is also important to believe that you can solve the problem.

Quick Draw is designed to develop powerful mental imagery, which will come in to play in both numerical and geometric settings and to encourage students to recognize that there is more than one way to solve a problem. When a figure is shown, a variety of interpretations are possible and learning that other persons see it differently can be liberating. That is, students can come to believe there is not just one way to solve a problem but that there are many ways of doing a mathematics task. Students who hold such a belief will be more likely to explore alternatives and act meaningfully than a student who believes there is just one way to do the task. This latter belief can be quite debilitating. As students are engaged in doing mathematics, they are likely to fear forgetting THE way to do the task and develop a level of anxiety while a student who realizes there are many ways to approach the task can be more confident, recognizing there are many paths and new ones can be created. Thus one of the goals of Quick Draw is to help students form the belief that problems can be solve in many ways and they have the capacity to construct their own way which makes sense to them.

Too often, students in mathematics class are carrying out procedures in a prescribed manner with little sense of what they are doing or what it all means. It may be completing a page of two-digit multiplication exercises using a taught procedure or applying memorized formulas in a mechanical fashion. Students may feel that they are supposed to listen carefully to the teacher's directions and explanations and then follow the procedure specified. This can result in a rather mechanical mode of functioning. Such an explain-practice method of teaching is ineffective and can actually be debilitating.

Once students come to believe that mathematics is memorizing and applying rules, it is very difficult for them to respond meaningfully to tasks which require decision making. The nature of the mathematics tasks students are asked to do in school has a profound influence on future success in mathematics requiring abstraction.

In Quick Draw, a figure such as the one shown below is presented briefly to students. They are asked to "Draw what you saw." The past tense saw is used because the students must make their drawing using mental images they have constructed since the figure is not observable while they are drawing. A discussion of their interpretations and methods of drawing is an integral part of the Quick Draw activity. An example of how a discussion might flow is described in the sample dialogue which follows.

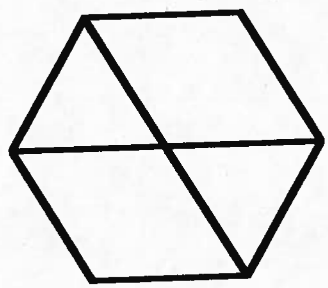


Figure 1.

Sample discussion (See Figure 1)

T:	What did you see?
Katrina:	I saw a hexagon with an X in it.
Rosie:	I saw two diamonds and two triangles.
T:	Please explain. Will you come to the overhead and point out the shapes you described? (She does)
Dorie:	It looks like an envelope opened up.
T:	What do you mean? Does anyone understand what Dorie is saying?
Pat:	I see it now. (goes to front of the room). These are the flaps. If they were folded down it would be an envelope.
Others:	Oh, I see it now.
Juan:	It looks sorta like a cube but a line is missing.
T:	I don't see that. Come up here and show us what you mean.

Juan: See, if there was a line right here, it would be a cube.

Christy: Oh yeah, I see it how!

T: Isn't this interesting. I show a geometric figure and you came up with so many different ways of seeing it. I was impressed with your thinking. We'll do some more of these another day.

Each individual gives meaning to their experiences in their own unique way. Some will see it as a two-dimensional figure while others may give it a three-dimensional interpretation. Individuals respond meaningfully to this task because there is no way to act otherwise. The individual must act - they are asked to draw what they saw. It is impossible to be passive. When they draw, they must work from a re-presented image since the figure shown is no longer in view. Finally, they are asked to describe what they saw and explain how they drew their sketch. As students listen to the ways others saw the figure, they are stimulated to reflect on their constructive activity and to consider other interpretations. Often, students will describe new ways of viewing the figure as a direct result of listening to the descriptions of others.

A second Quick Draw shape is shown in Figure 2. When this figure was shown, Danielle, a fourth grade student, drew a hexagon followed by the three interior segments. It was clear from her action and subsequent comments that she had constructed a vivid image of the figure shown and she drew with confidence. It is also clear that she "saw" a two dimensional figure. This drawing previously had been used with many children as well as adults and most of them reported seeing a cube. Some children and even adults may at first have difficulty making a drawing satisfactory to them. Note that the manner in which this activity unfolds, each person has a chance to be successful. Some will be satisfied with their drawing after the first exposure. Others will not have their drawing finished until after the second look while other students will draw it only after it is exposed for discussion but everyone has the opportunity to complete a drawing without their efforts being judged by anyone.

In the National Council of Teachers of Mathematics Curriculum and Evaluation Standards, developing spatial sense is emphasized. Quick Draw is designed to develop spatial sense. According to the Standards, a general goal of school mathematics is making sense rather than just memorizing and practicing procedures.

QUICK DRAW DIRECTIONS

Quick Draw is designed to develop spatial sense, encourage the transformation of self-constructed images and develop geometric intuitions through discussion.

Quick Draw should be used with the whole class. Overhead trans-parencies can be made from the blackline masters provided. Each pupil should have unlined paper and pencil. You may have them draw horizontal and vertical lines to divide the paper in fourths so they can make four drawings on one sheet. Show the first design for about three seconds. Say, "I am going to show you a shape for three seconds and I want you to build a mental picture you can use to draw what you saw." Avoid the temptation to show it for a long period of time - - it is important that students work from imagery rather than copying what they are seeing. Say, "Draw what you saw." The students should have their pencils down during the presentation. At first you may get complaints about not having enough time. After a few moments show the shape again. Show again briefly if you feel it is necessary. This will only be necessary for more complex figures. Three times is usually sufficient and two times is the norm. When students seem to be finished with their drawings, show the shape on the overhead so students can compare their drawing to the actual picture. With the design in view, ask, "What did you see?" At times you may want to follow-up with "How did you draw it? What did you draw first?" This talking about mathematics encourages students to reflect on their imaging. Ask students to name the geometric figures they see. Geometric language will be used naturally. You may wish to supply mathematical names for such objects as trapezoids and parallelograms as needed by the students for communication. Much geometry can be learned through Quick Draw.

Follow the same procedure for each design. Present several shapes in a session. Be sure to use this activity initially for at least three days and then incorporate it with other topics throughout the year. This is an excellent activity with which to begin class. The figures in this book are organized in seven levels. There is a progression in complexity of the figures through the levels but there is considerable variability at each level. Some schools may wish to designate certain levels to be used at particular grade levels so that students will not see figures they have drawn before. Select a design to present using your judgment of the appropriate level of complexity for your class. Begin with simple designs and present increasingly complex ones. As a variation and challenge, have students draw the figures upside down or rotated quarter turn. Students also enjoy making up their own designs to be used in class. You will note broad individual differences in the students' initial drawings and descriptions.

As you use this activity, note the improvement in student's drawings and verbal descriptions. Also note individual differences. Some students may be very good at this task but not at arithmetic. Pay particular attention to the students who excel at this task since they may have unrealized mathematical potential.

QUICK DRAW

Quick Draw helps students develop

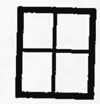
1. mental imagery
2. recognition of shapes
3. analysis of mental images
4. spatial memory
5. concept of symmetry
6. geometric vocabulary
7. negotiate social norms

<u>TEACHER MATERIALS AND PREPARATION</u>:

1. <u>Three</u> or <u>four</u> Quick Draw transparencies.
2. Overhead projector (OH)
3. Blank paper for covering

<u>STUDENT MATERIALS:</u>

1. A pencil
2. unlined paper divided like this:



Whole class

"I will show you a shape for only a few seconds. Try to make a mental picture so you can draw it after I turn off the projector. Ready? On the count of three, ONE, TWO, THREE."
Turn on OH. Show the line pattern for 3 seconds. Turn off OH.
When most people have drawn all they can, prepare to show it to the students again.
"I will now give you another look. Ready?"
Show for 3 seconds.
When most people have drawn all they can, turn on the OH and leave it on. Some students will be able to draw it looking at the shape when they could not draw it otherwise.

↙

Whole class discussion

The discussion of their drawings is the heart of the activity.
"What did you see and how did you draw it?"
Encourage students to talk about their drawing. Do not rush the discussion. Let it continue as long as new ideas are being put forth. Some students will be inspired by what others say. It is not unusual for five or more different ways of seeing the figure to be described.

Imagery and Mathematics Learning

The student who can construct and transform images is likely to be successful in doing mathematics.

There is compelling evidence that imagery plays a significant role in mathematical reasoning (Reynolds, 1993; Sfard, 1994). For example, a young child using a compensation strategy for adding 7 + 5 may think of "moving" one from the 7 to the 5 to form 6 + 6, a known fact. Or a child determining how many one inch cubes there are in a rectangular solid 3" by 3" by 4" may visualize the solid as composed of three layers. Whether working in a numerical or geometric context, when students are engaged in meaningful mathematics rather than rote computation, it is quite likely they will be using some form of imagery (Brown and Wheatley, 1989, 1991, 1994; Reynolds, 1993; Sfard, 1994). There is also compelling evidence that mathematicians use imagery in powerful ways (Hadamard, 1949; Nunokawa, 1994; Sfard, 1994). Mathematics is not just a logical subject but is laden with imagery. The word imaging is used as a metaphor for mental activity we cannot yet fully explain in terms of neural functioning.

While engaged in mathematical activity, whether of a numeric or geometric nature, students construct images. For example, they may be shown a geometric figure briefly and asked to draw what they saw. When they make their drawing, they are operating from a constructed image. The nature and quality of the image will influence the drawing which results. If at a later time they are asked to draw what they saw, the students then re-present the image.

When an image is constructed, only certain features of the experience are included (Kosslyn, 1983) . In Piagetian terms, the nature of the image results from the way the experience is assimilated. The nature of the image is dependent on prior mental constructions, intentions, and the situation under which the image is constructed. For example, an image of an isosceles triangle which has a horizontal base and a vertex angle above the base. If this image of a triangle is the child's only image of a triangle, then their concept of triangle is quite limited. Children have a richer concept of triangle when they can transform their image of triangle flexibly.

What an individual constructs depends on his or her mental schemes (images) formed from prior experiences. The line drawing shown in Figure 2 was presented briefly to a class and they were then asked to draw what they saw (Yackel and Wheatley, 1990). This figure was described as two squares, a small square and two trapezoids, a hallway, a skylight and a pyramid with the top cut off. Some individuals constructed an image of regions and others of joined segments, some two-dimensional and some three-dimensional interpretations. Even though the same figure was presented in the same manner to individuals, the nature of the images constructed varied greatly.

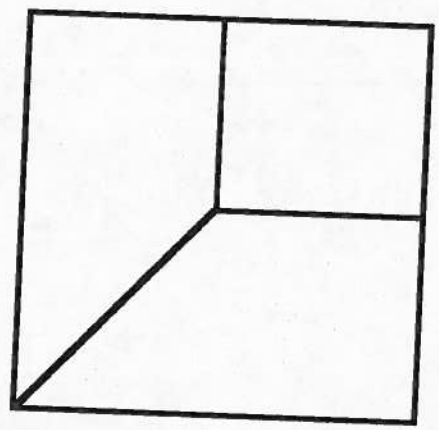


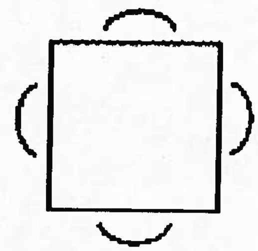
Figure 2. Quick Draw figure.

Sfard (1994) has reported that mathematicians who feel they have a deep understanding have constructed some abstract image which makes the knowledge into a whole - an abstract image. This level of knowing is often metaphorical in nature. Thus it appears that pictures in the mind become part of image-schemata (Dorfler, 1991; Johnson,1987) and then a number of image-schemata coalesce into a higher level (abstract) image. For communication purposes it may be useful to refer to pictures-in-the-mind as basic images and others which are formed from image-schemata, abstract images, keeping in mind that there are many levels of abstractness.

As an example of imaging in problem solving, consider the Long Table Problem. which we have used with fifth grade students.

The Long Table

Tiffany is arranging tables for a party. She has 12 square tables which she wants to put together to make one long table. Each of the small square tables seats one person on a side. How many people can be seated?



Most students successful in solving this problem elaborated their image of one table to twelve, often drawing out the twelve tables, and proceeded to count how many seats were available. One student made no marks on paper but explained he had a mental picture of the twelve tables in a row and he could count the number of seats available. This student had powerful mental imagery. The individual differences in imaging among children is striking.

A good example of the role of re-presentation occurred in the following task (Brown and Wheatley, 1989). We gave fifth grade students 36 Multilink cubes and asked them to construct a "solid" rectangle. Students began this task in a variety of ways. For some, it was clear that they started with an image of a rectangle: as they worked they kept the opposite sides of their shape of equal length and worked in a systematic fashion. Some students began by making rods of the same length from the Multilink cubes and joining them together. Others formed a rectangle of cubes with no plan for the number of cubes on each side and began placing Multilink cubes to fill the interior. In most cases the lengths chosen were not such that the remaining cubes just filled the interior and they began all over again. They saw the task as trying to use all the cubes in a way which was consistent with their image of a rectangle. Some students used their image together with the number of cubes given to determine the size of the rectangle they made. However, for other students, it was clear they started with no useful image of a rectangle. The task for these students became a very different one. Many of them were not successful in constructing any regular geometric figure with the cubes. One student constructed a five by seven array of cubes with one left over; she was unable to modify her array to use 36 cubes.

Reynolds and Wheatley (1994) describe a fifth grade girl's solution to the problem, "On a Rubik's Cube (3x3x3), how many small cubes have exactly two faces showing?" While at first Elaine answered 24 (four on each of six faces), once she looked at a Rubik's Cube she quickly revised her answer and confidently said 12. The nature of the image constructed was crucial in her solution. Her first image of a cube as composed of six faces is a frequently reported image. The second is more sophisticated and, in this case, more relevant to the question posed. Her initial image of a cube from which she had obtained 24 was six uncoordinated faces. Her subsequent image of a cube was three layers of nine small cubes each.

Summary

A thesis of this book is that meaningful mathematics learning is frequently imaged-based. While there may be certain forms of mathematical reasoning which seems to not use imagery, most mathematical activity has a spatial component (Reynolds and Wheatley, 1992). If school mathematics is procedural, students may fail to develop their capacity to form necessary images of mathematical patterns and relationships. It is well documented that students who reason from images tend to be powerful mathematics students. Further, it is asserted that being able to use images effectively in doing mathematics can be developed. When students are encouraged to develop mental images and use those images in mathematics, they show surprising growth. All students can learn to use images effectively. However, it must be acknowledged that there are great differences among individuals in their ability to form and use mental images. Some people inherit the ability to image while other have less capacity. Having said this, it must also be acknowledged that with appropriate experiences everyone can greatly improve their spatial sense. Thus school mathematics programs should make improving spatial sense a priority.

(See references at the back of this book.)

Learning from Students' Drawings

In designing activities for students, it is important to infer what the students know. Using our knowledge of students thinking we can choose tasks which are challenging but possible. Much can be learned about students' imagery and conceptualization by studying their drawings made in a Quick Draw setting. Their drawing activity provides a window into their minds. Students' activity is greatly influenced by their concepts and images. While we as adults may "see" a cube in the figure below, to some children it may be the shape made by three Pattern Blocks.

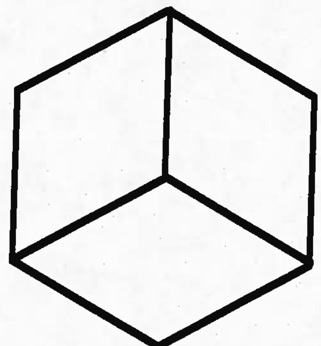


Fig. 3. Quick Draw shape.

Students do not just copy a figure using a photographic image of it. The image constructed is constrained by what the child knows. If a child's concept of triangle is a shape with a horizontal base and two segments slanting down to it, she may have difficulty drawing a right triangle.

Examples of students' actual drawings made in a Quick Draw setting are presented and discussed below. You are encouraged to analyze the drawings your students make.

Fig. 4. Joan's drawing and QD shape shown.

Students draw what they "see." As shown in Figure 4 above, Joan drew two triangles because that is what she "saw" (mentally constructed). Rather than constructing an image of the whole figure, she partitioned it into two parts. It appears that Joan constructed the shape of a right triangle, since each of the triangles are nearly right triangles yet she did not construct the orientation of the triangles: there are no horizontal segments in her figure.

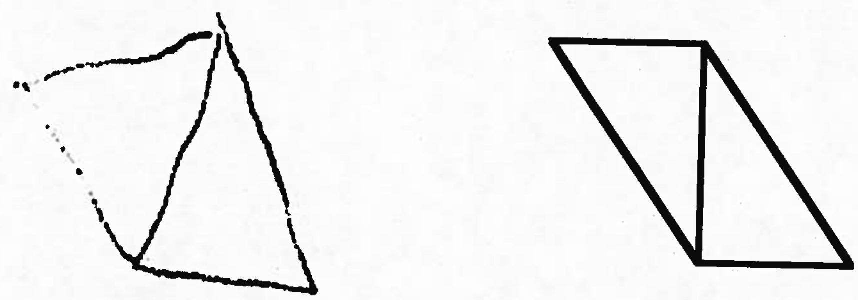


Figure 5. Mandy's drawing and QD shape shown.

It is very difficult for many primary grade students to draw a right triangle. Mandy drew two acute triangles with a common side (see Figure 5). There is no evidence in the drawing that she constructed any right angles, in fact, she most likely "saw" what we would call acute triangles.

Notice the path. She began at the top and drew the line down to the right, then the roughly horizontal base but, most strikingly, she continued up, slanting to the right, to complete her triangle. For her, triangles have sides slanting down to a horizontal base. The figure is completed by drawing two segments on the left. The top segment is not horizontal since again, she is drawing what she has conceptualized as a triangle.

This task has the potential of creating a perturbation, which can lead to a mental reorganization, a reorganization which may result in the conceptualization of a right triangle.

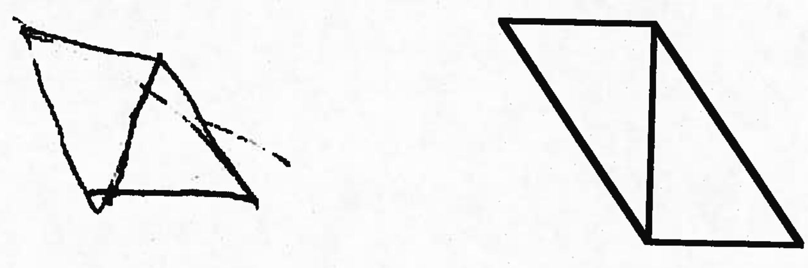


Fig. 6. David's drawing and QD shape shown.

The typical way of drawing a triangle with a horizontal base and two sides slanting down is quite dominating for many persons. David drew two acute triangles with no indication of any construction of right angles. The triangle on the right is nearly equilateral and likely is a depiction of his concept of triangle. The triangle formed on right is the "normal" orientation of a triangle. The top side of the triangle on the left is not horizontal as it was in the figure shown, perhaps because, for David, the triangle is "upside down." Also there was no attempt to form a vertical segment - triangles were constructed and the vertical orientation of this segment was not part of the mental construction.

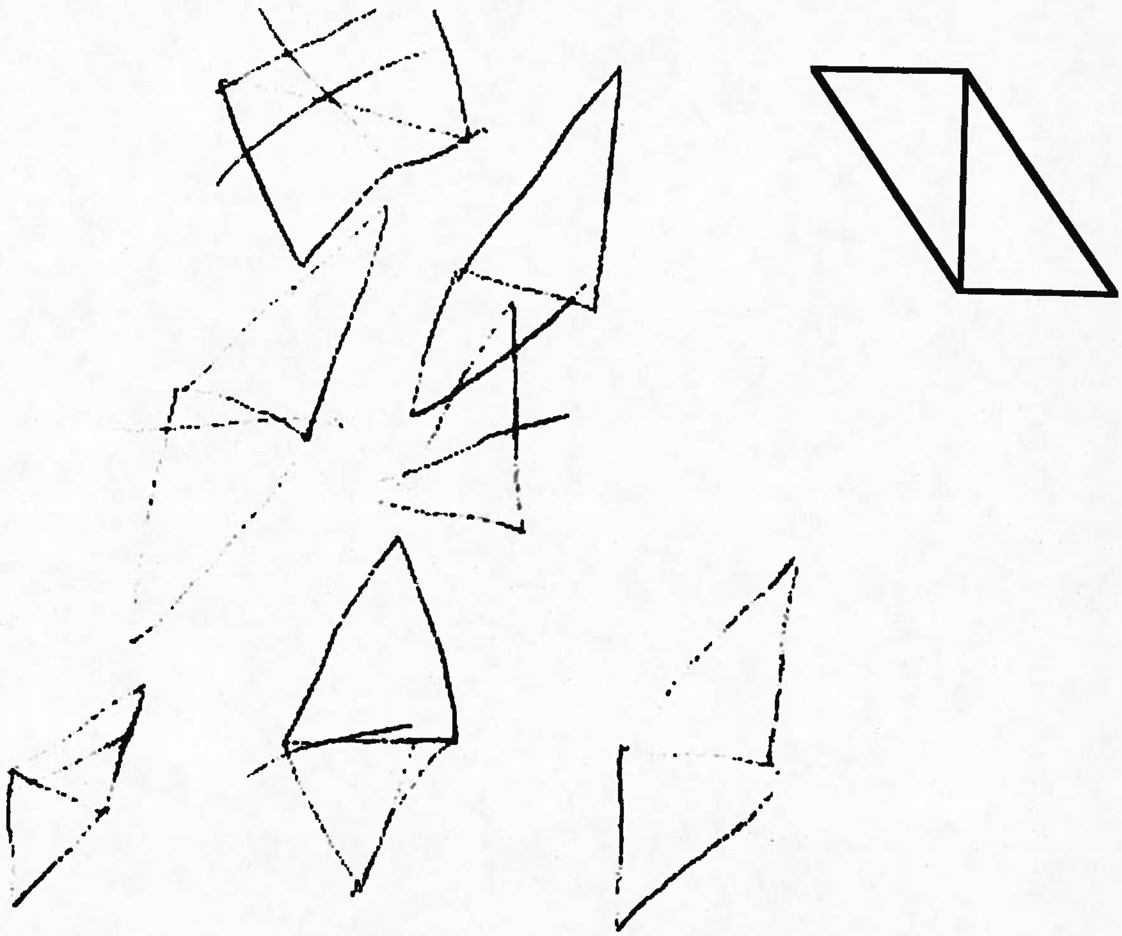
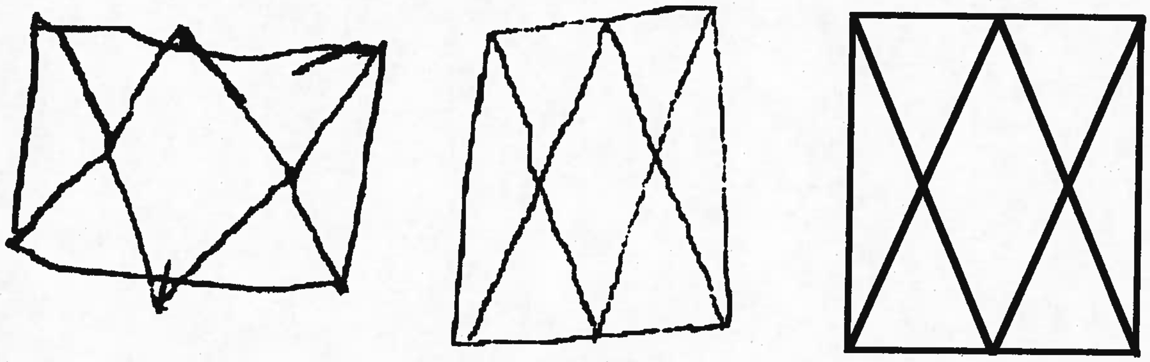


Fig. 7. Series of drawings and QD shape shown.

The set of sketches in Figure 7 shows the repeated attempts of a student to "Draw what he saw." We can see the strength of the conceptualization of a triangle as "Two sides slanting down to a horizontal base." The fact that this student made six attempts at drawing the shape suggests dissatisfaction with previous attempts; there was a perturbation. Note that the one figure on the left which is not crossed out is a break-through effort. A right angle is formed in the lower triangle. This set of drawings shows the power of Quick Draw to influence children's thinking. The final drawing on the right has right angles even though it is a reflection of the shape shown. Did the child "see" the figure reversed?



Christy Sheila QD shape shown

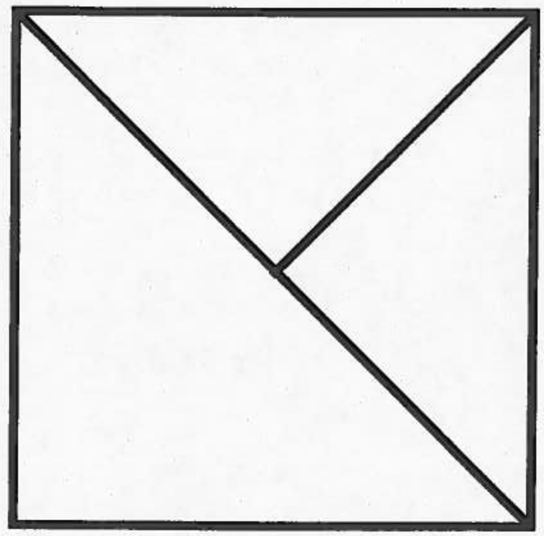
Fig. 8. Two drawings and QD shape shown

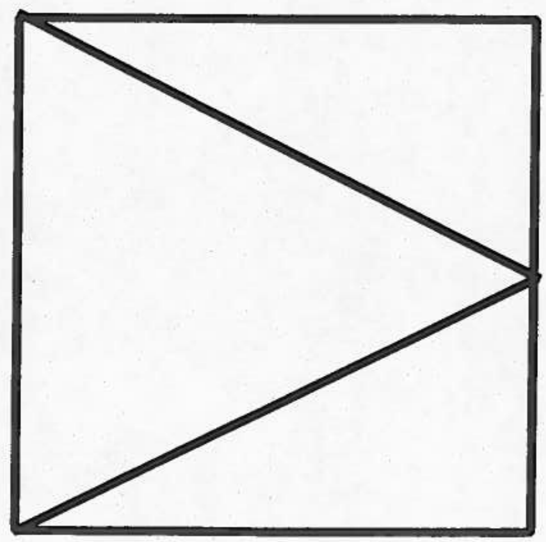
Some students form a holistic image of the figure shown while others see many parts. Judging from the drawing, it seems that Christy saw many distinct segments (See Figure 8). She drew a diamond and attempted to form the shape with separate segments. Rather than seeing an integrated design, she saw the diamond and attempted to make the shape by working from it. However, she did form a mental image of the figure which she used in making her drawing.

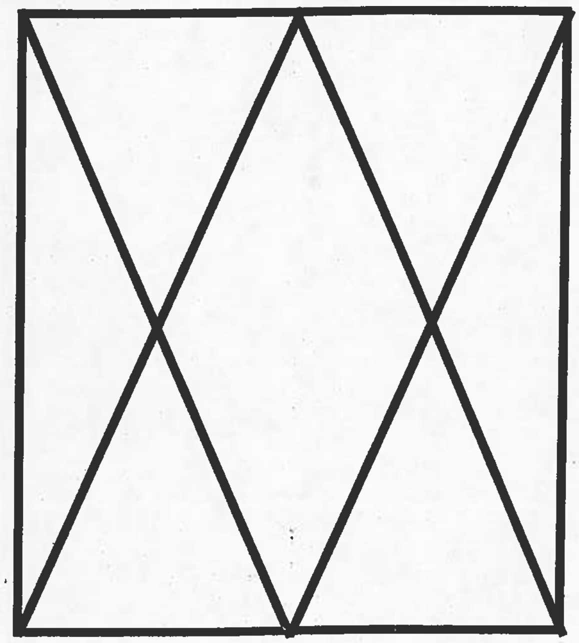
In contrast, Sheila saw the relationships of the parts to the whole. She drew the rectangle in one motion, then drew the upside down V and then formed the V with two segments. Sheila's imagery was quite strong. Her drawing showed the general shape as well as the relationship of parts. Other students have described this figure as 1) a rectangle with two Xs in it, 2) four triangles making a diamond in the middle, and 3) two triangles in a rectangle.

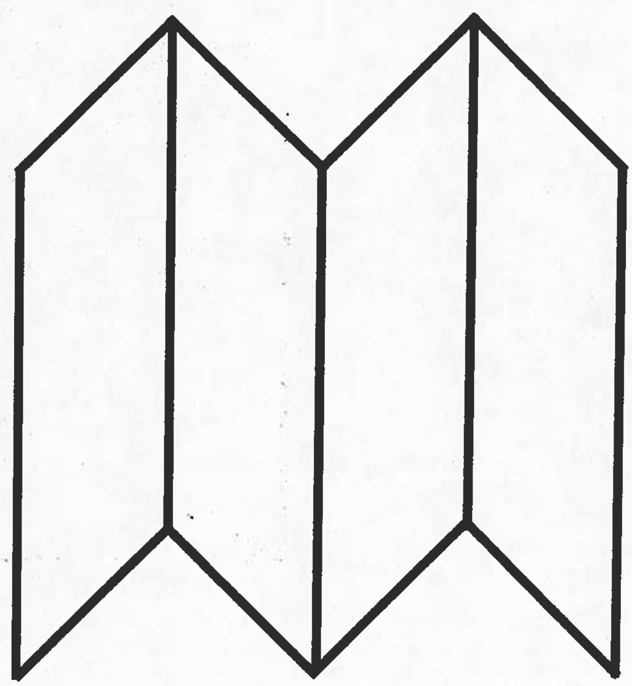
LEVEL 2

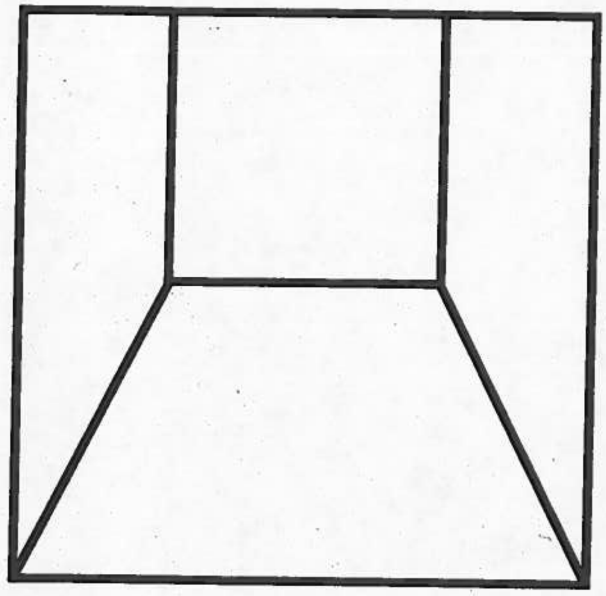
QUICK DRAW SHAPES

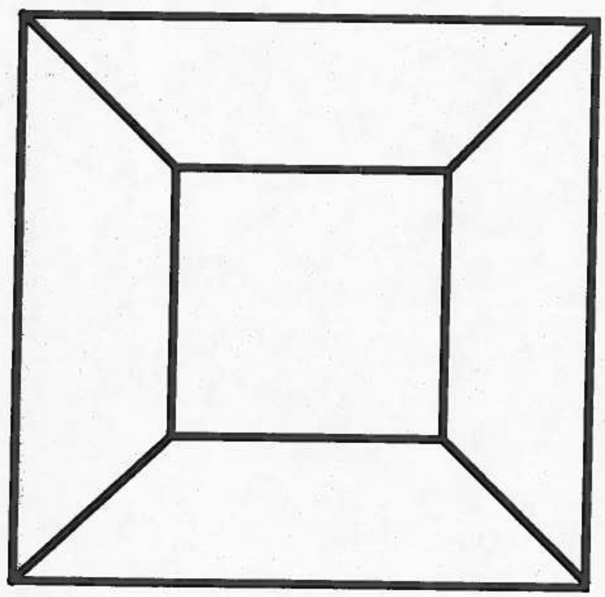


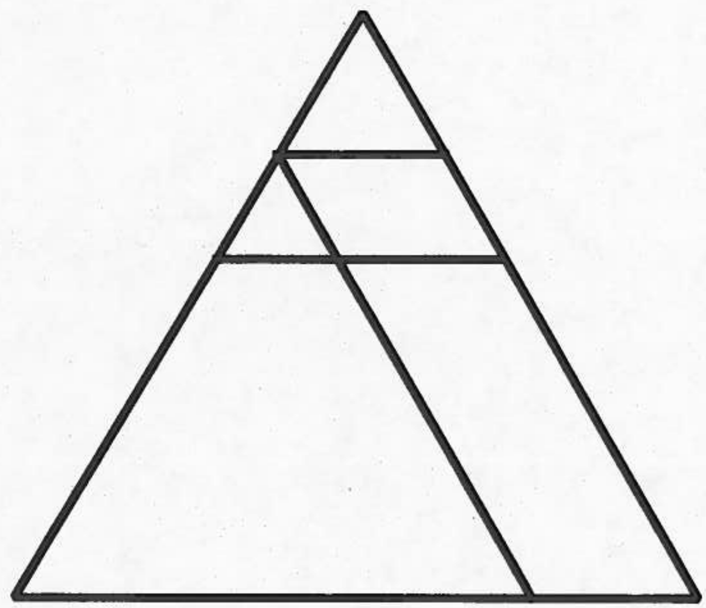


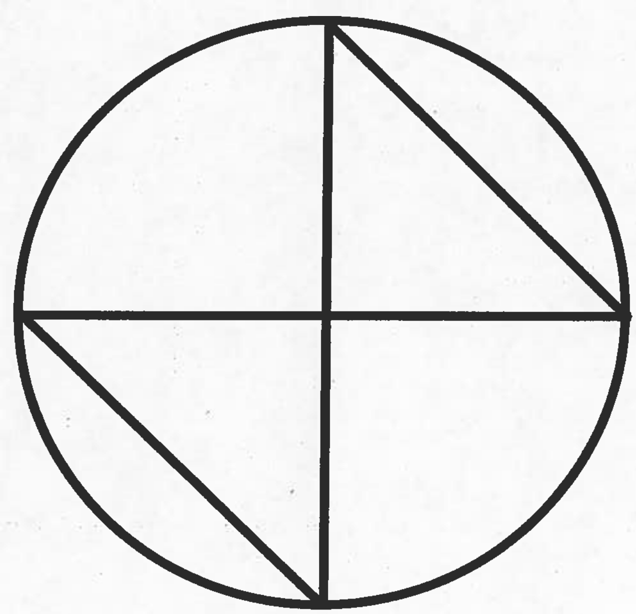


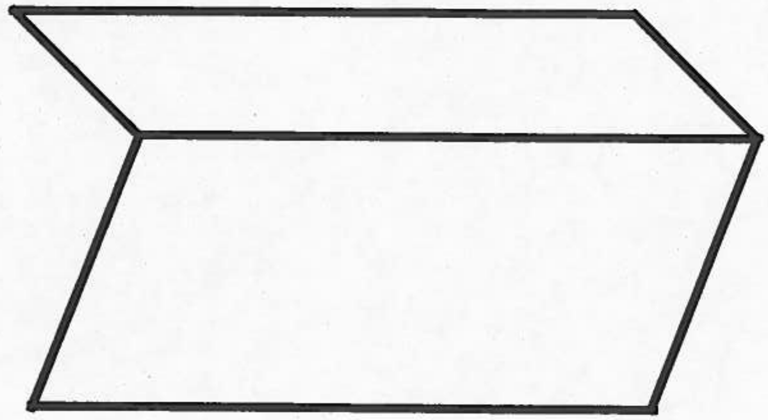


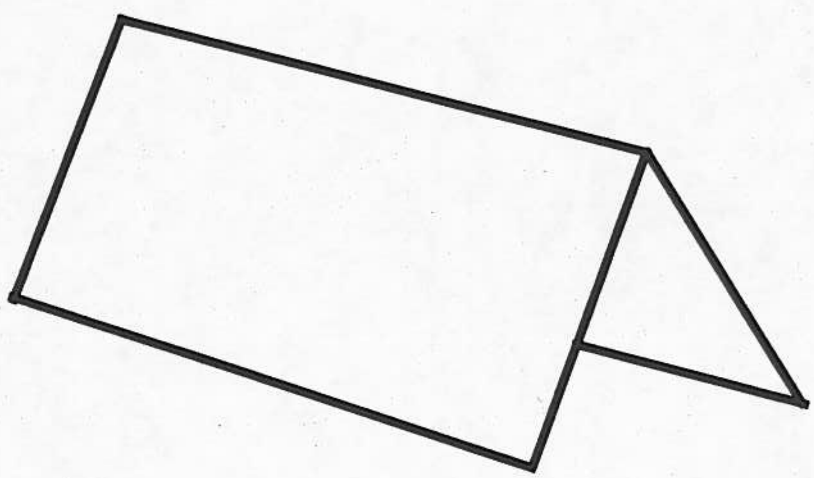


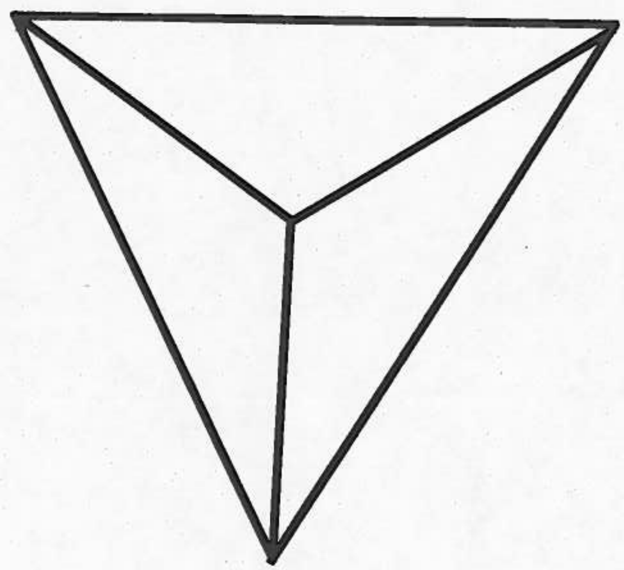


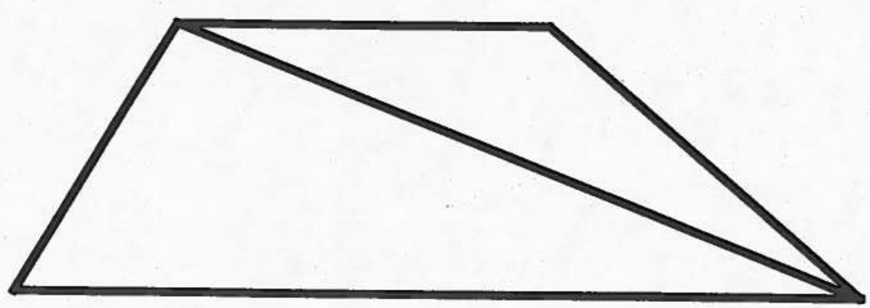


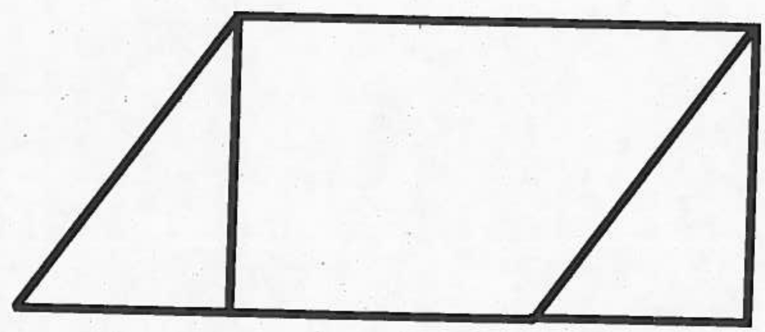


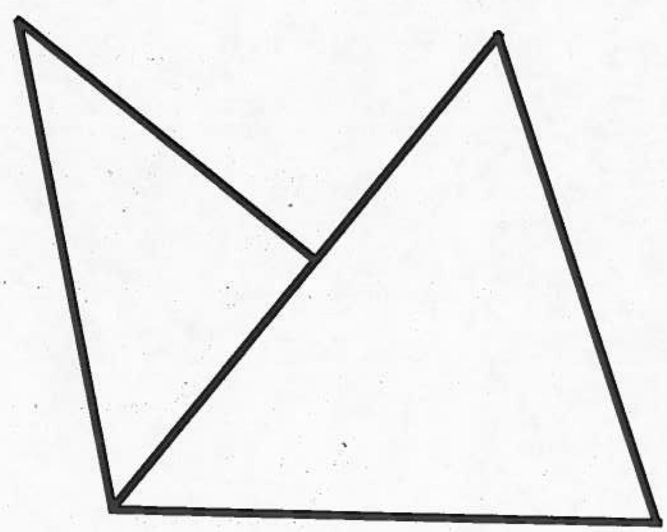


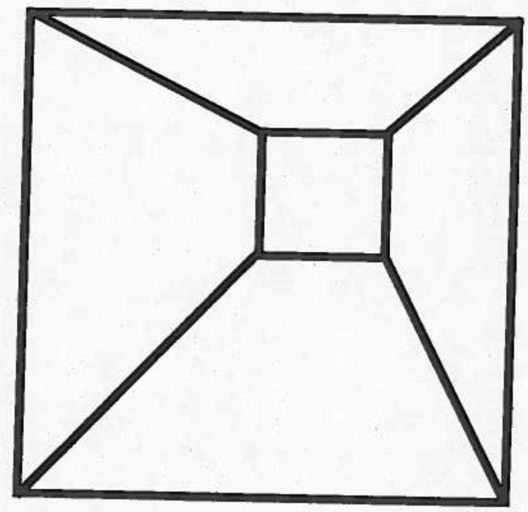


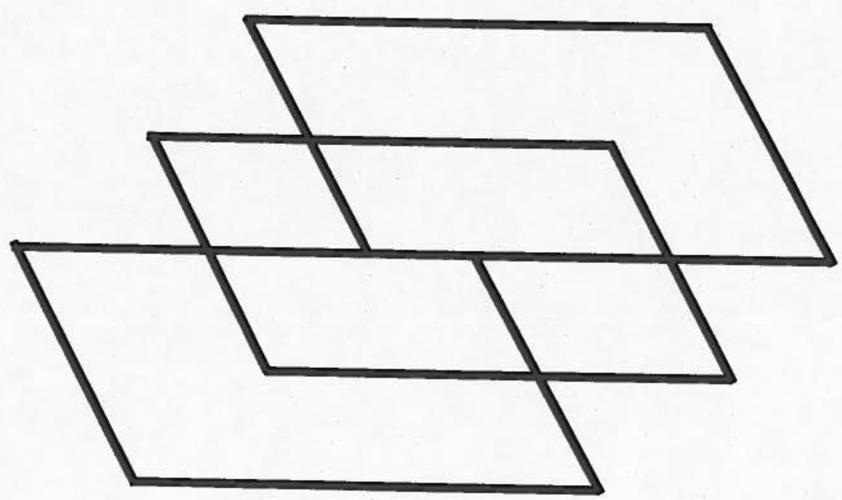


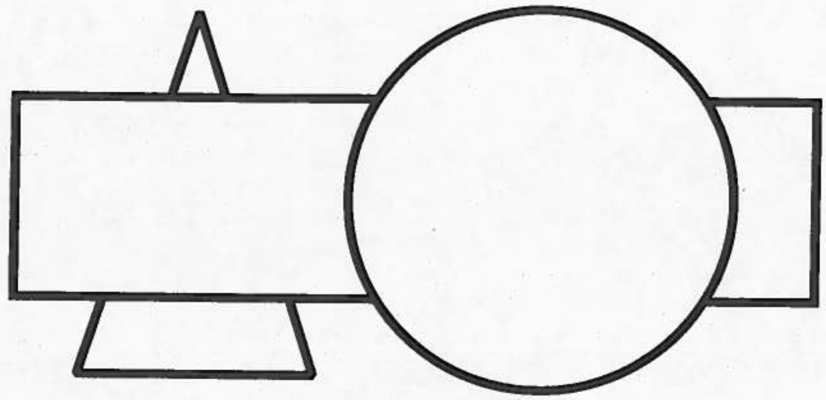


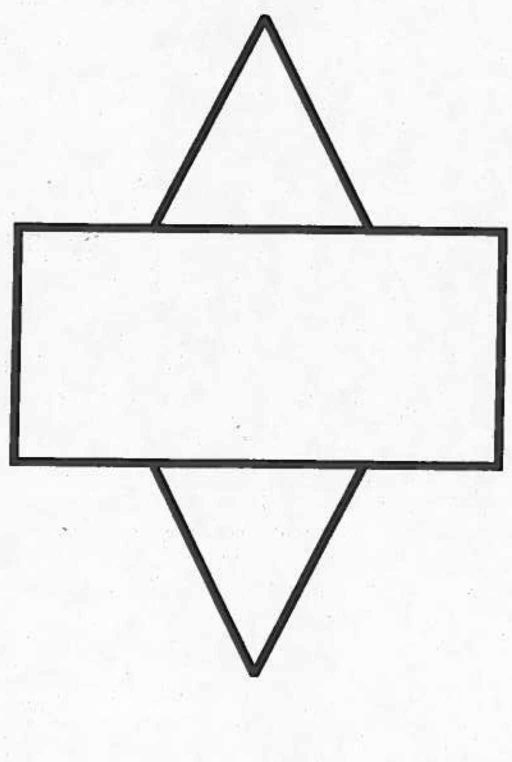


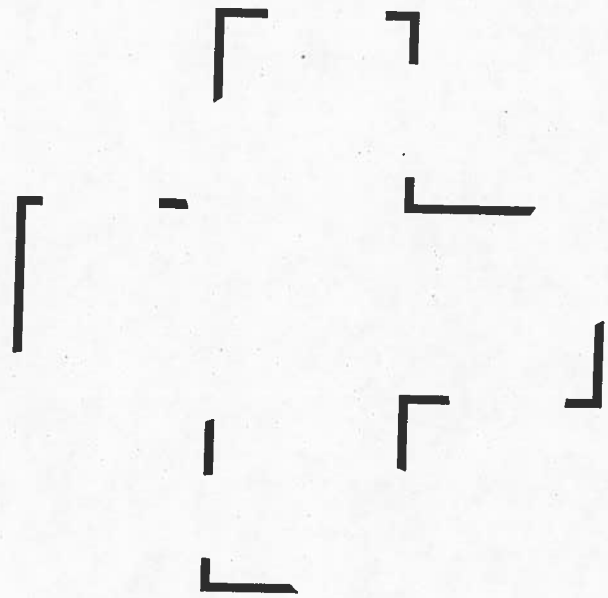


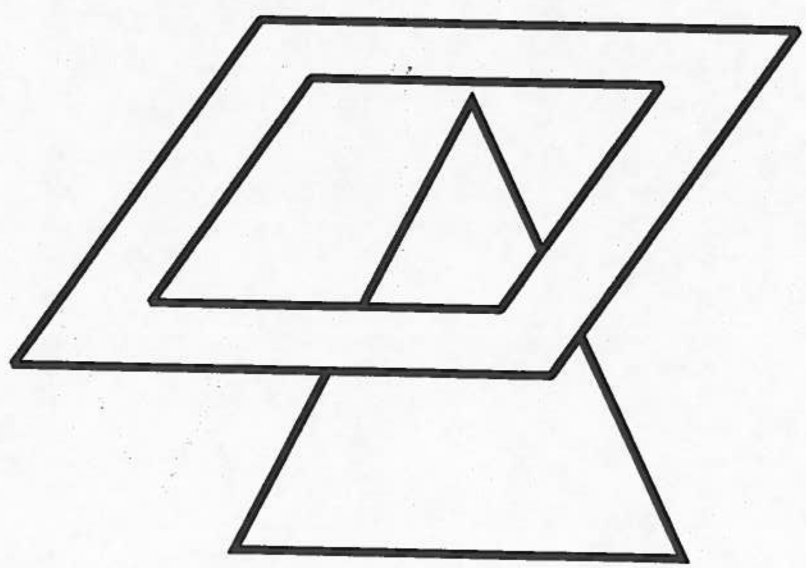


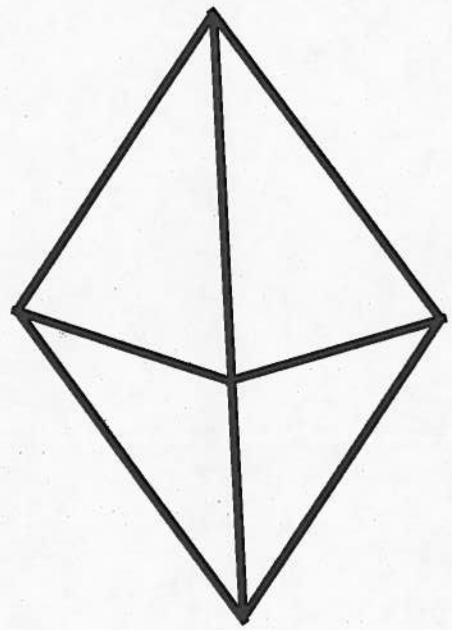


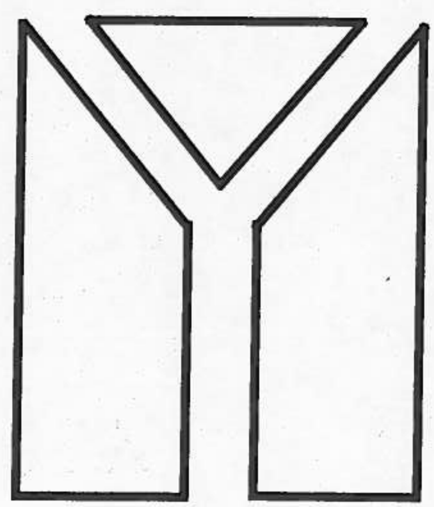


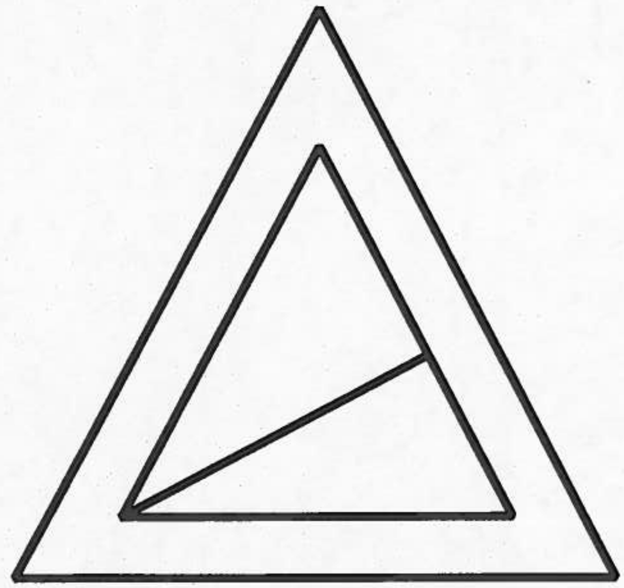


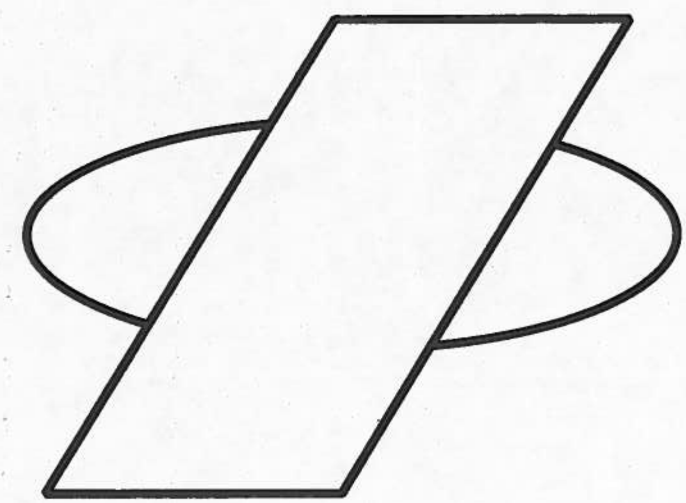












LEVEL 3

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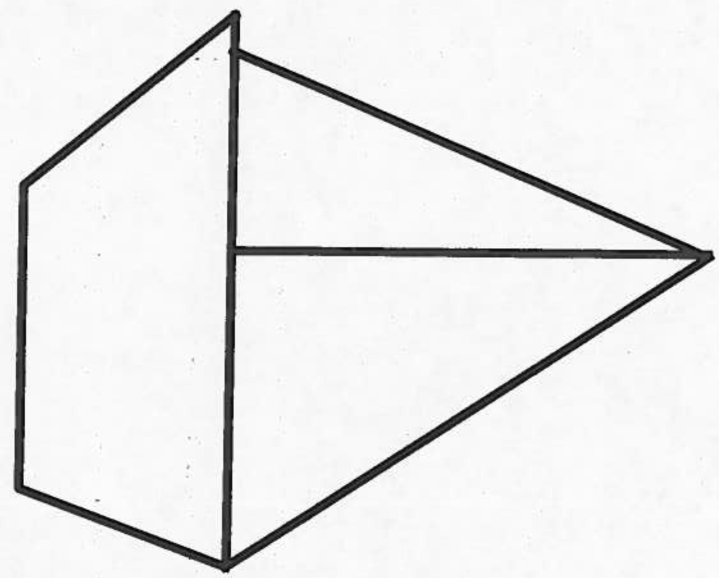

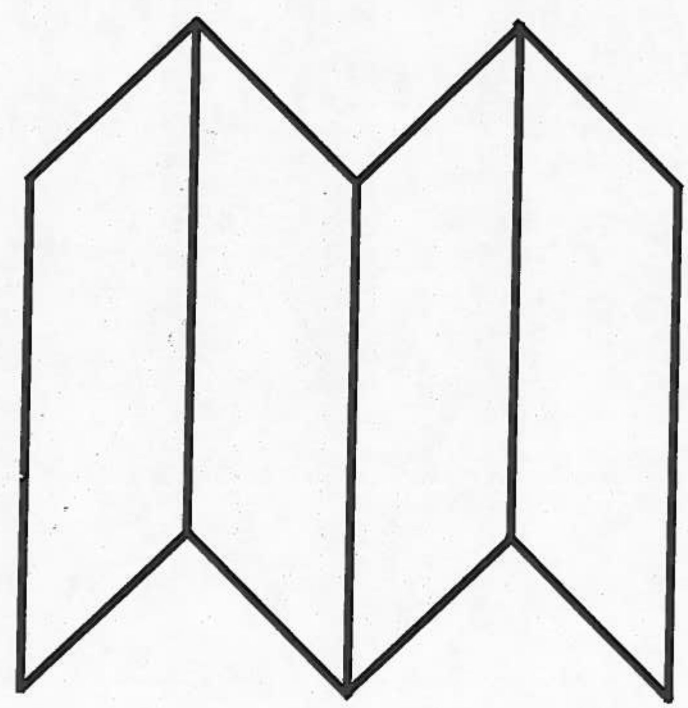

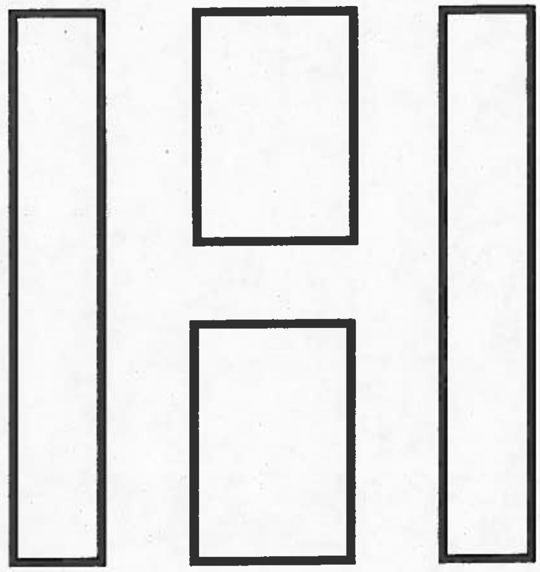

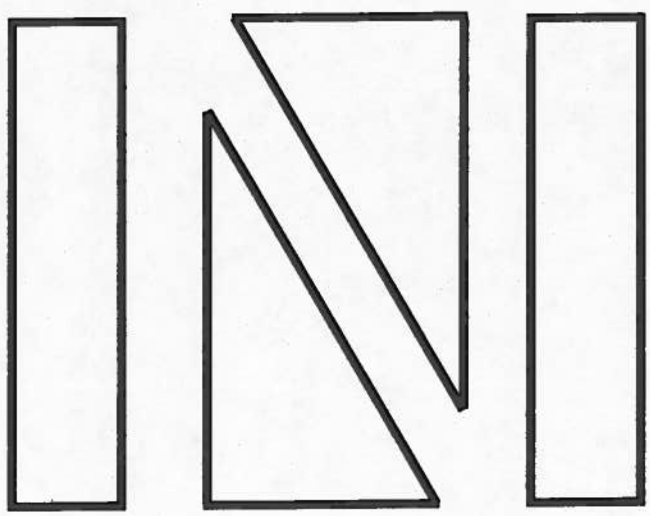

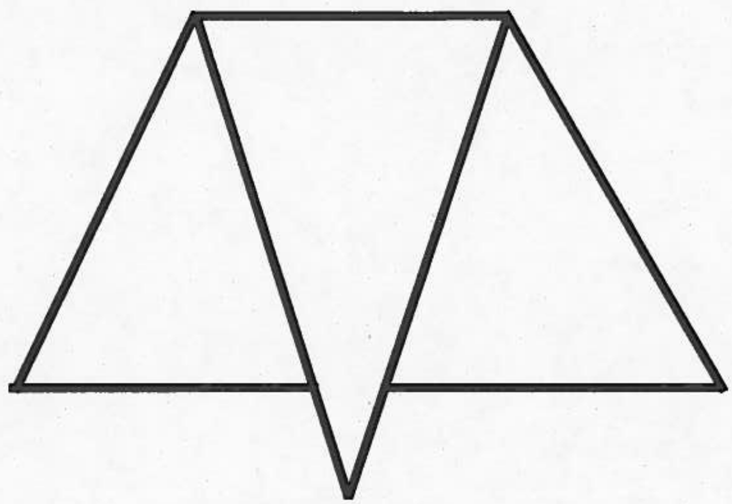

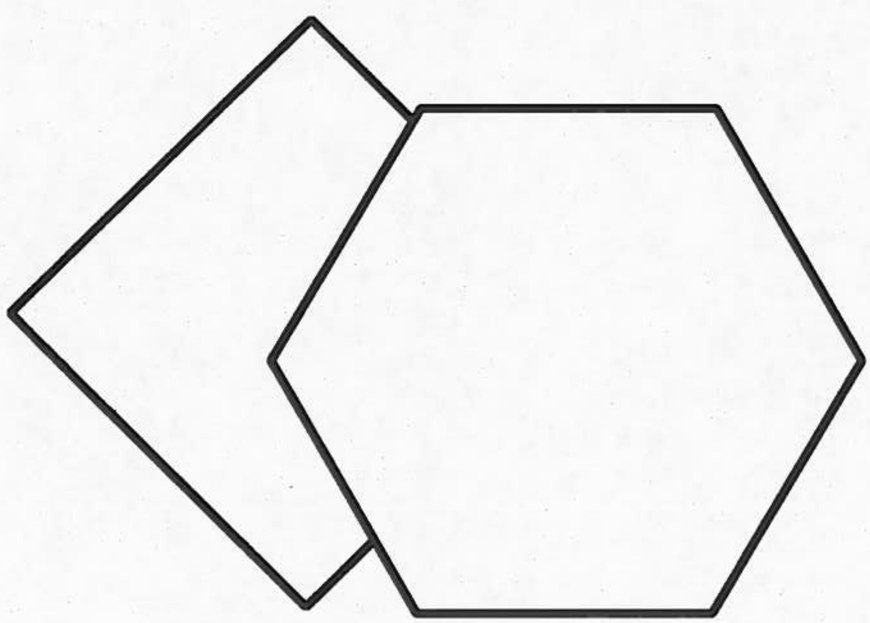

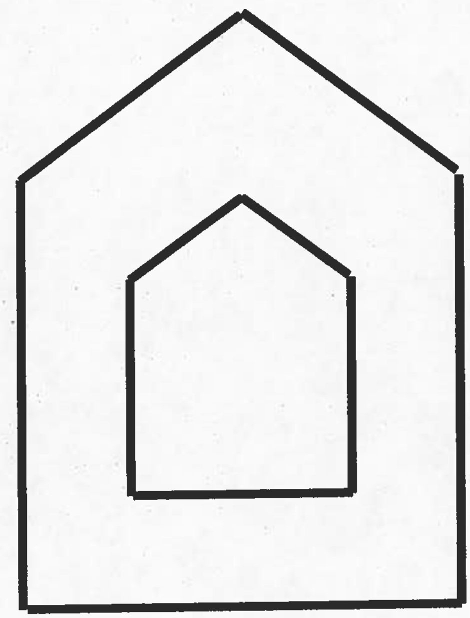

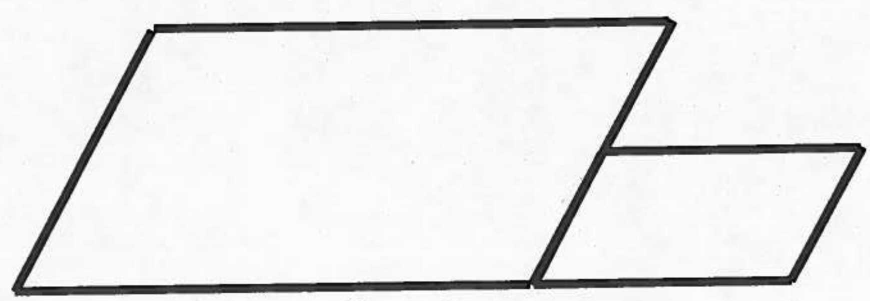

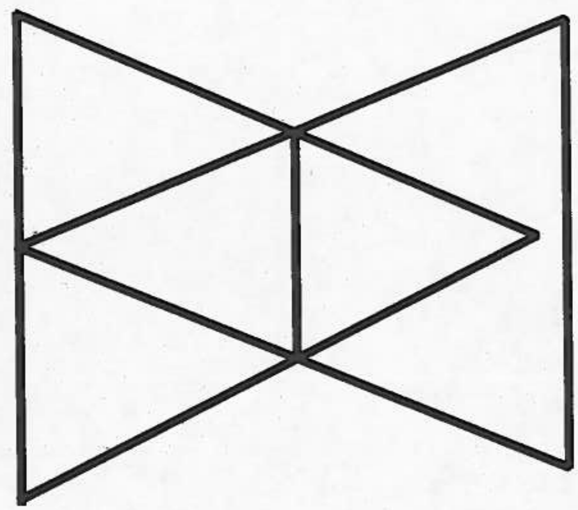

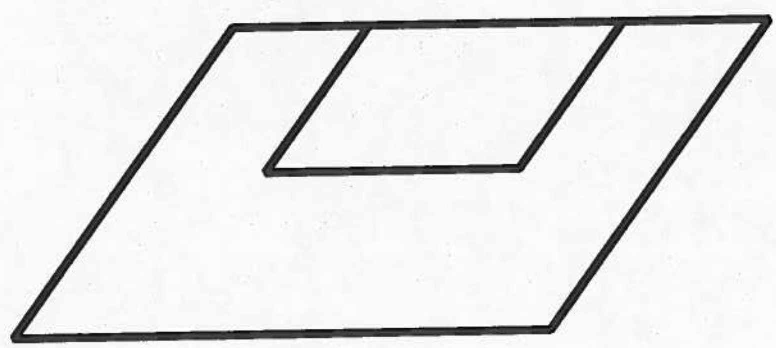

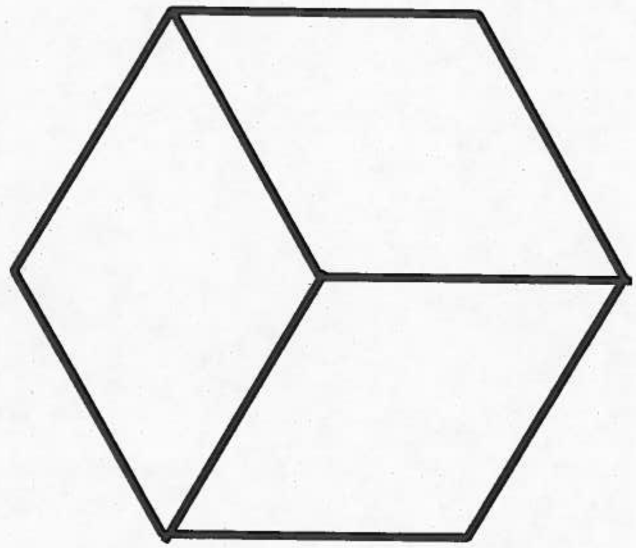

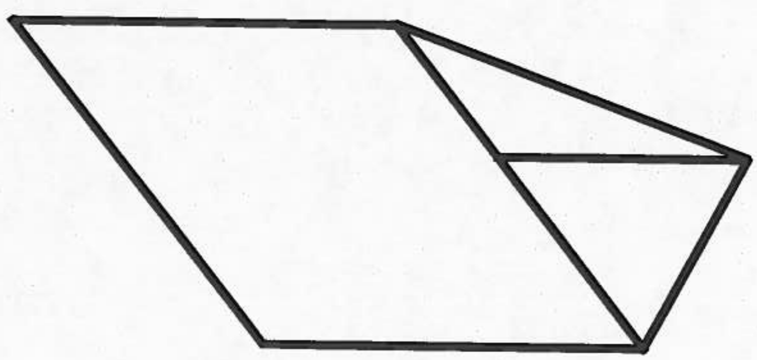

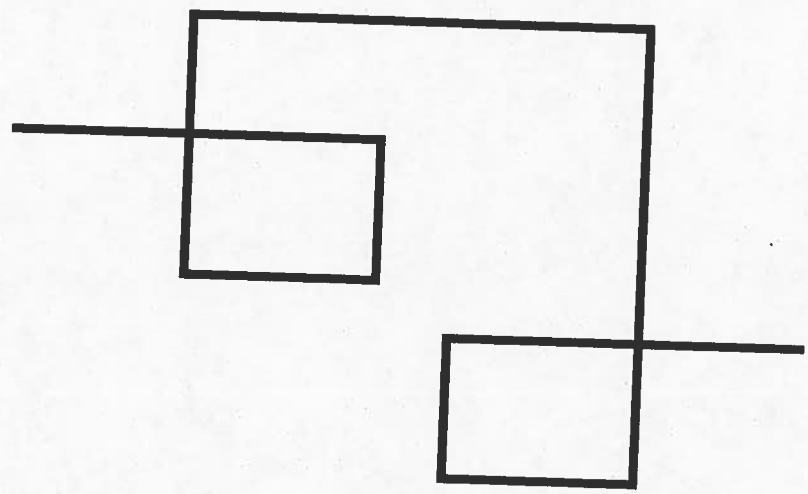

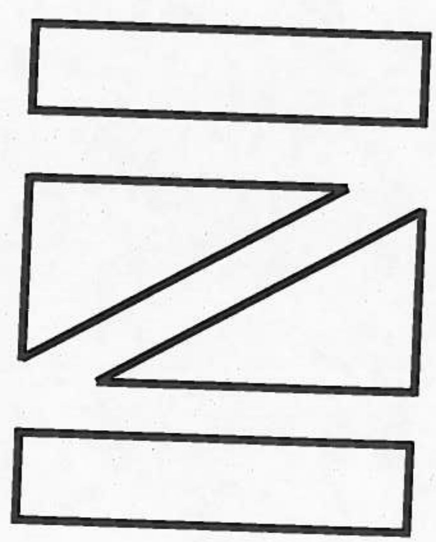

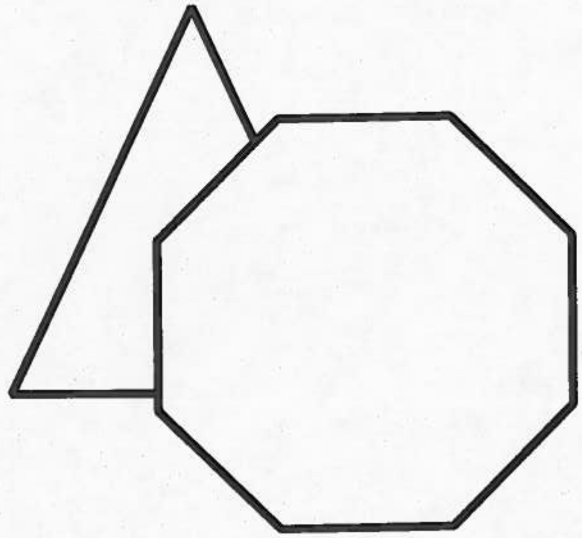

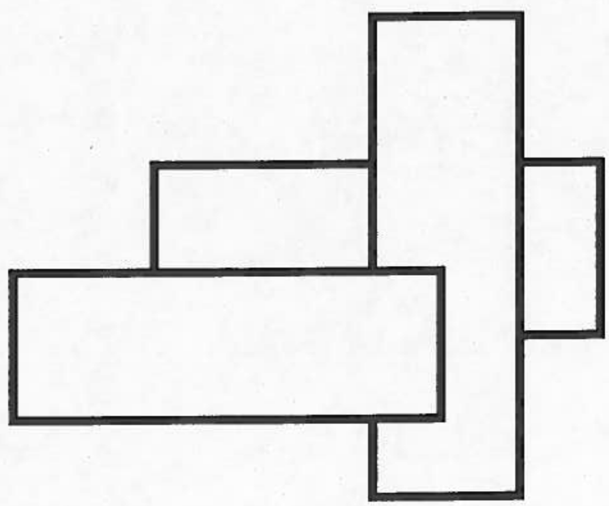

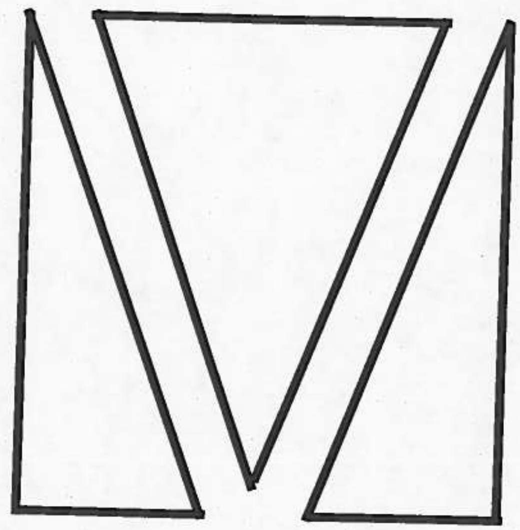

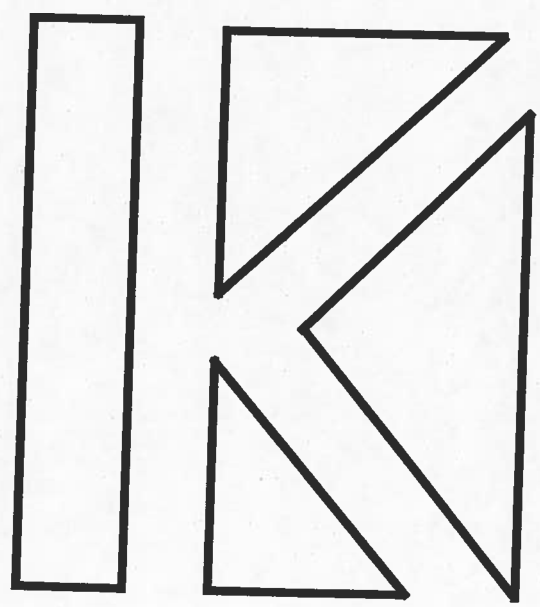

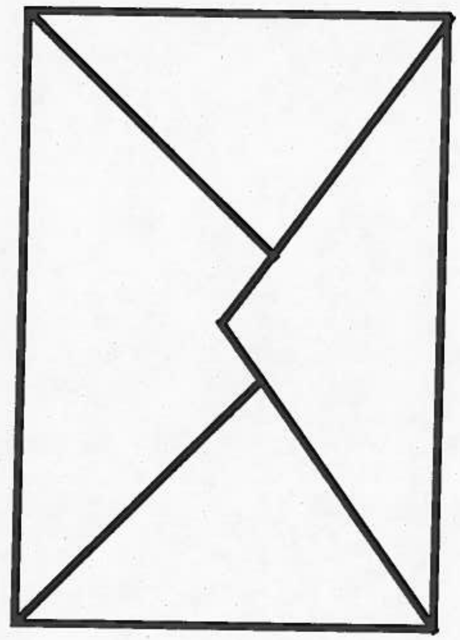

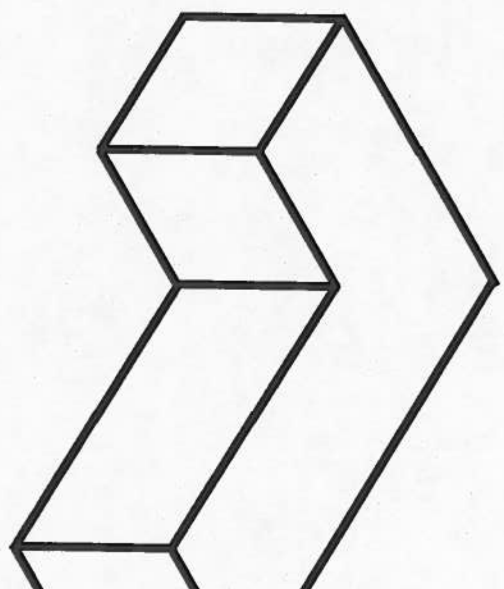

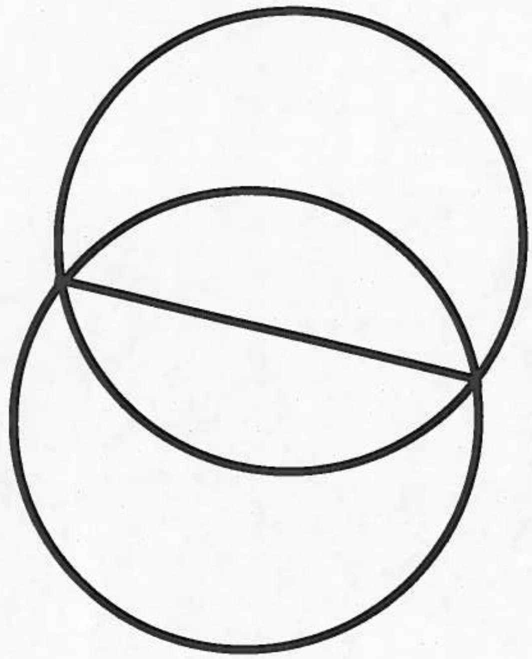

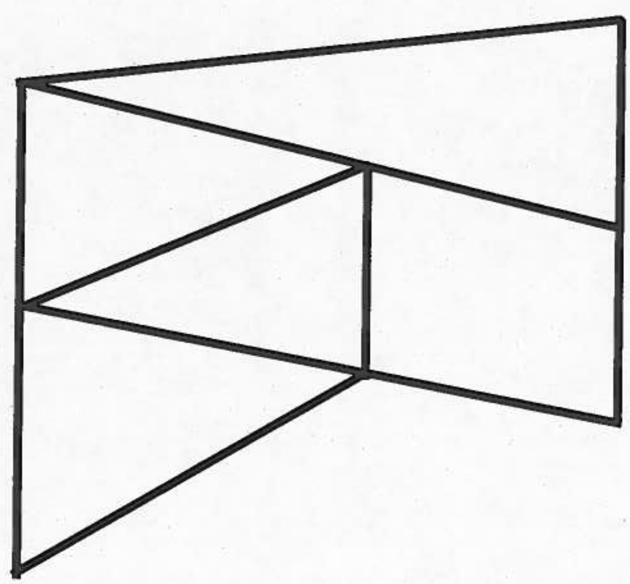

LEVEL 4

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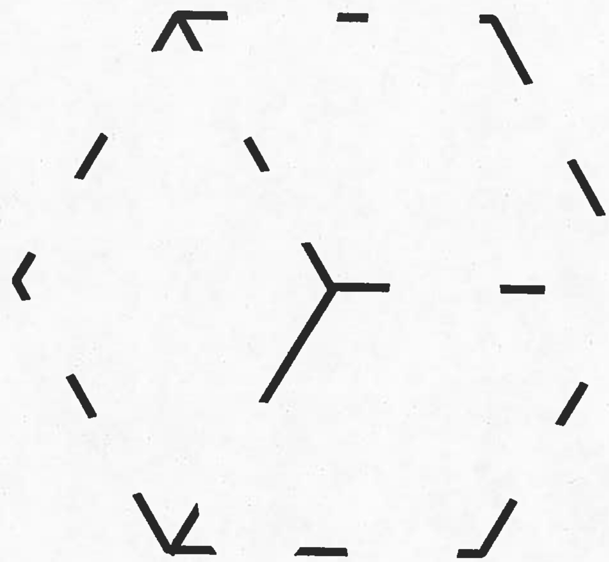

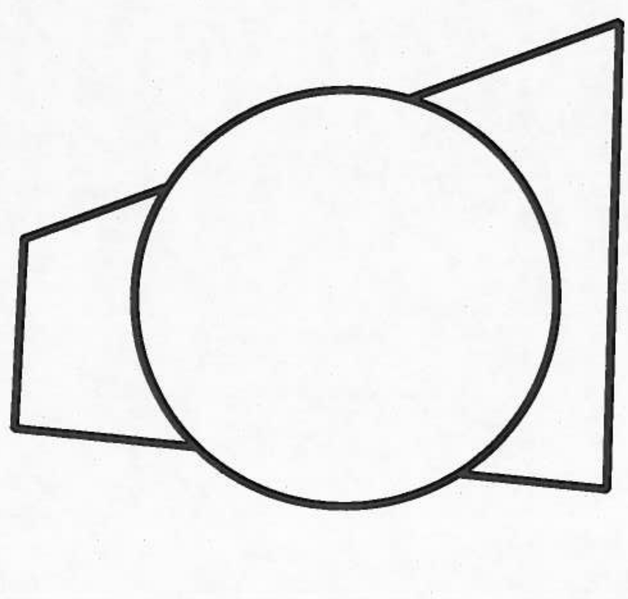

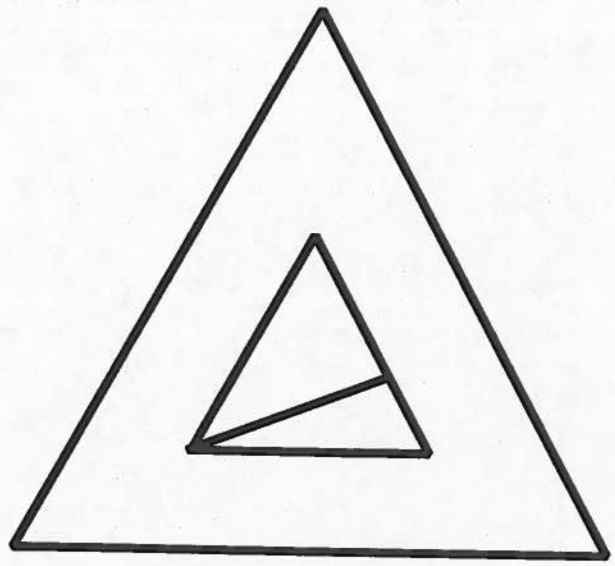

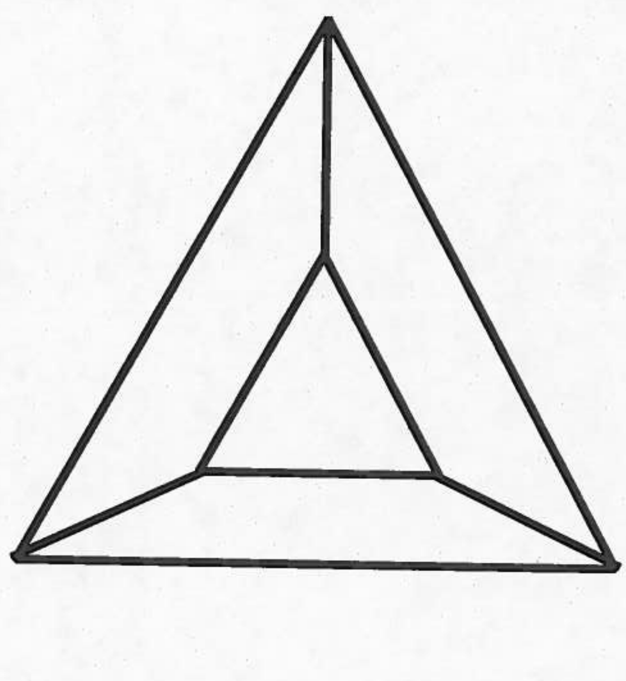

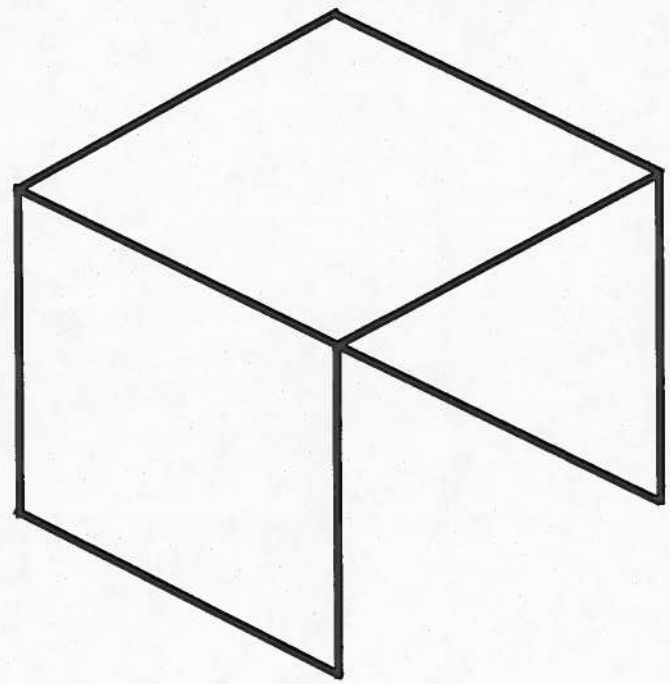

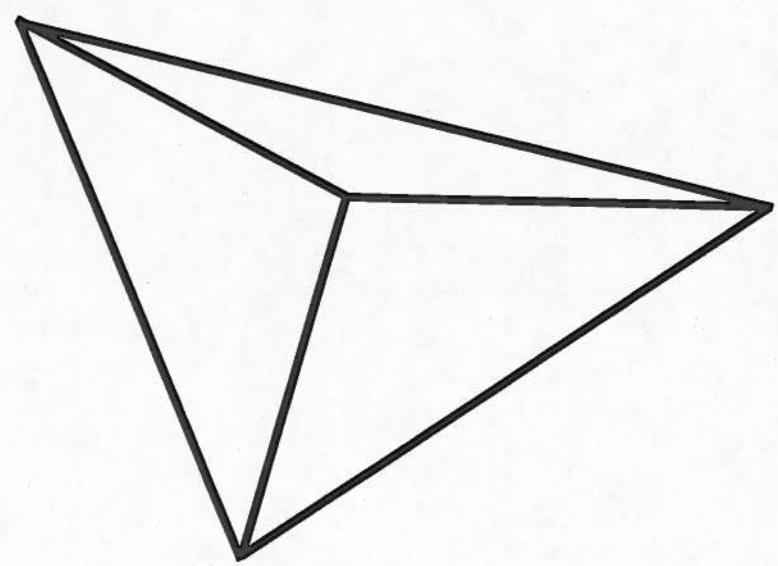

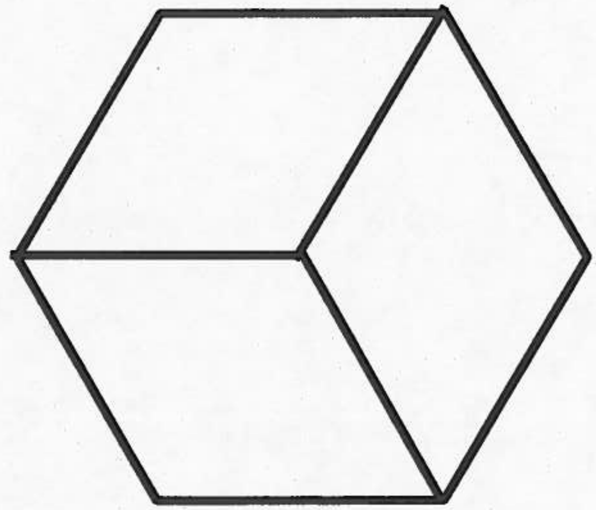

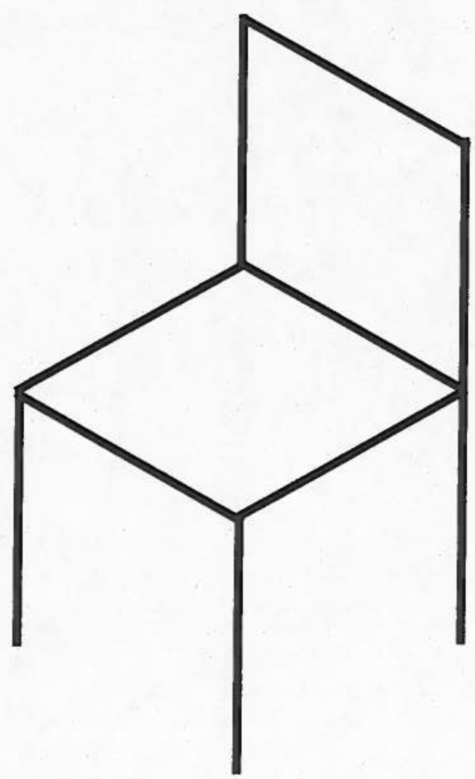

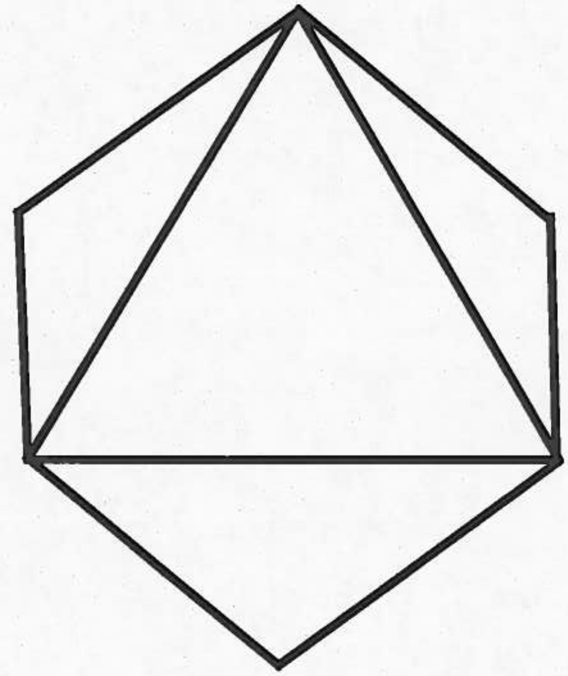

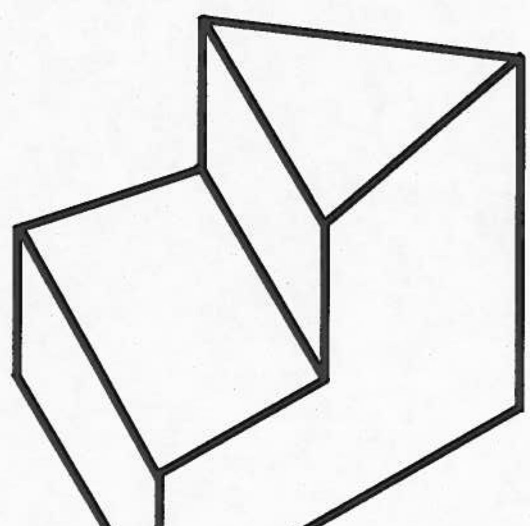

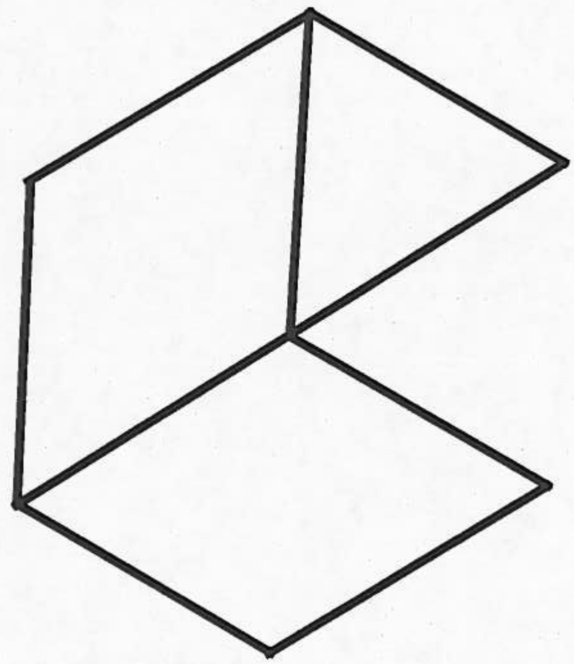

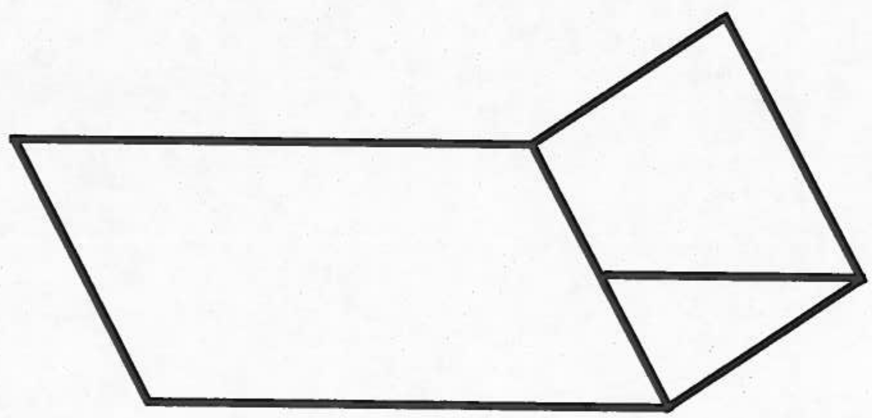

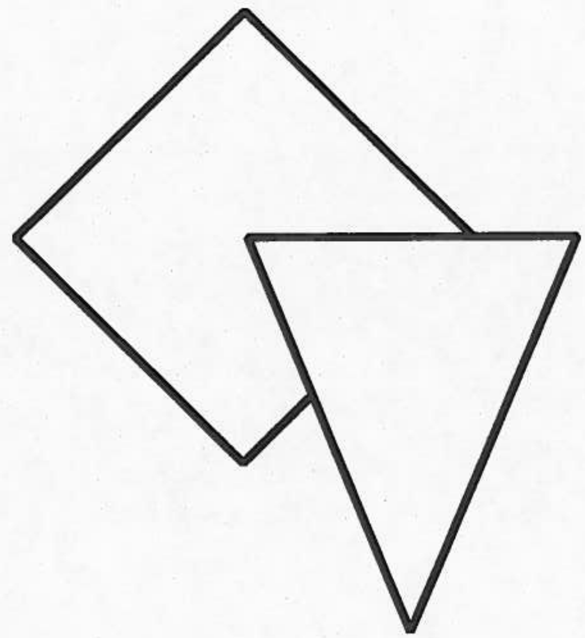

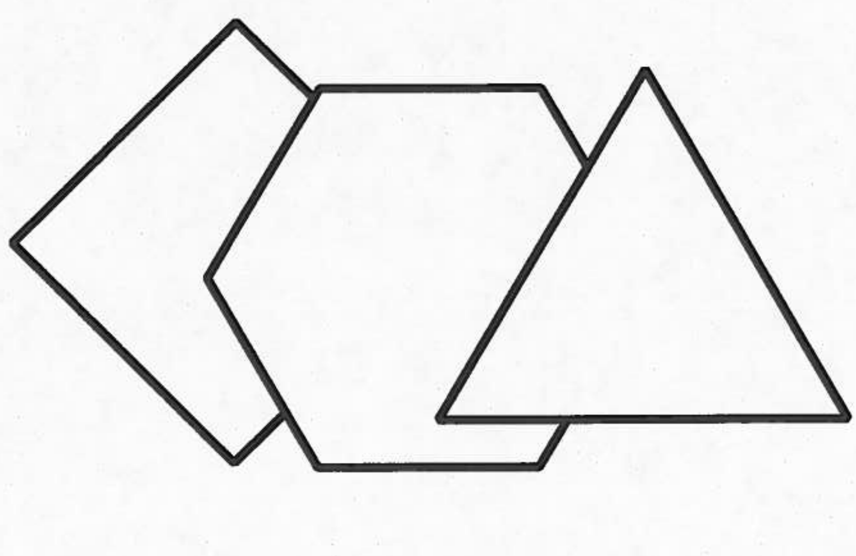

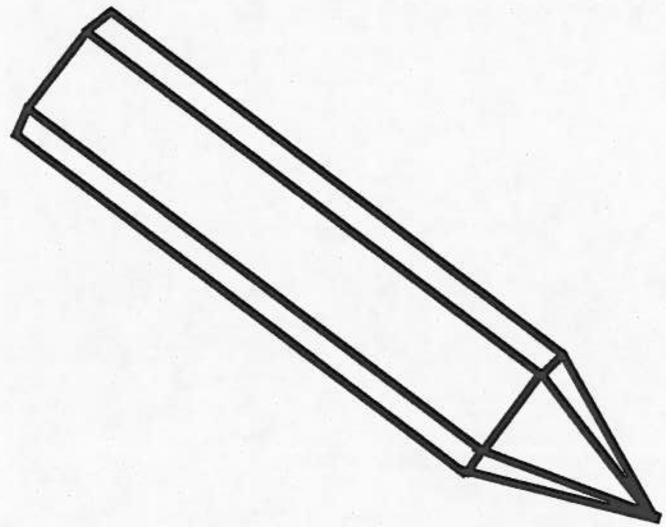

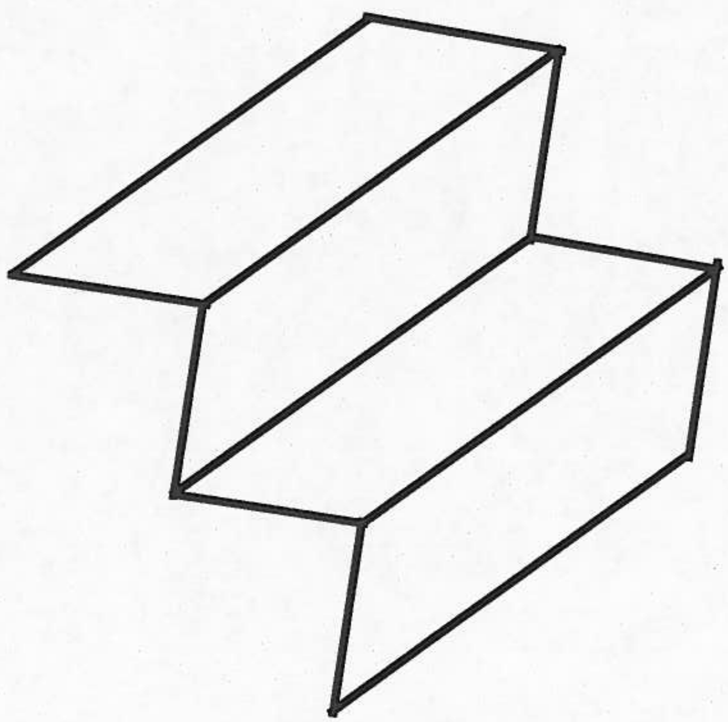

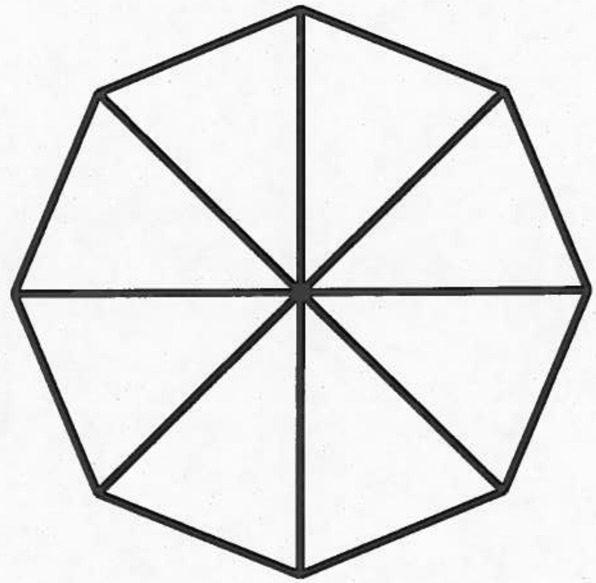

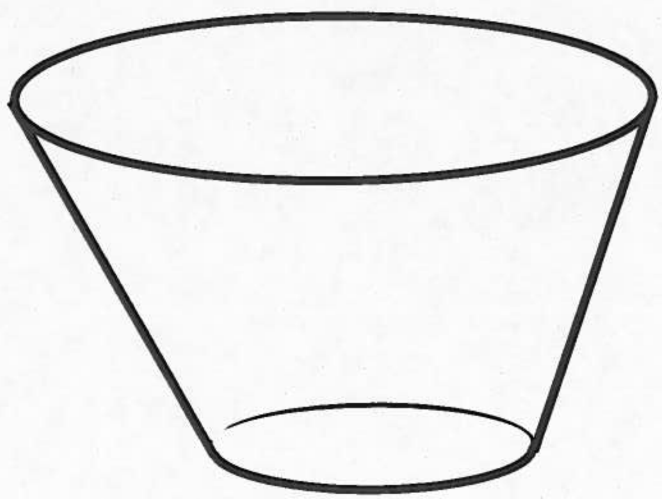

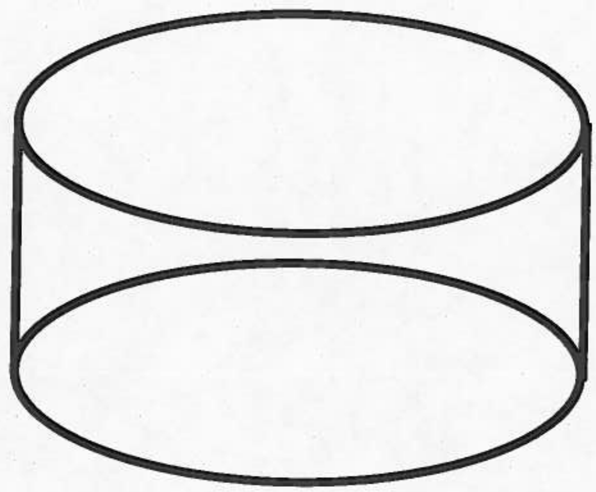

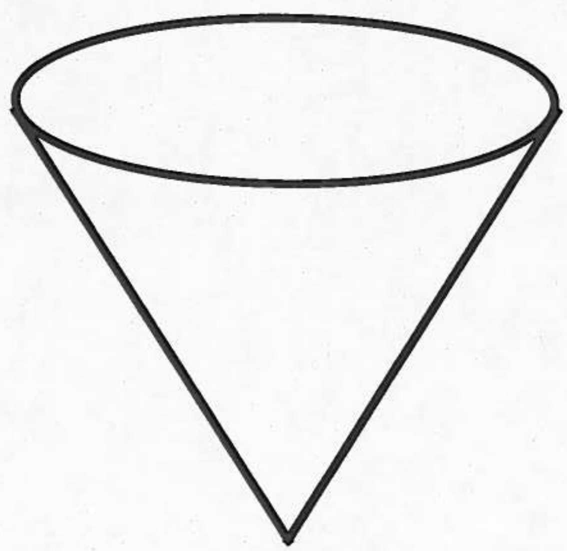

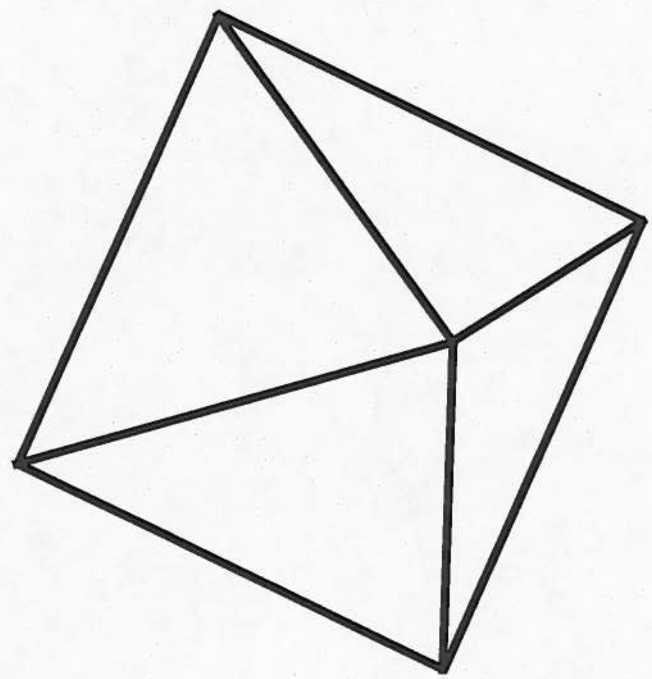

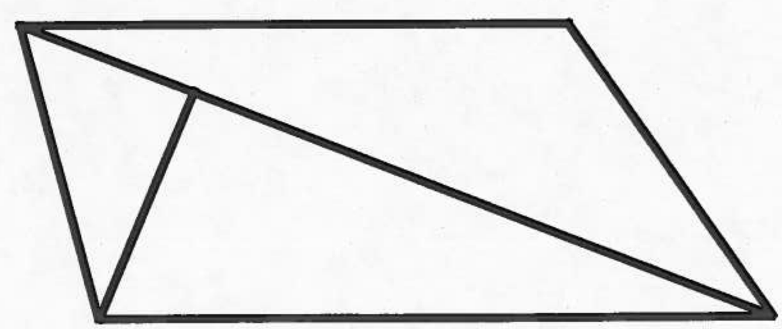

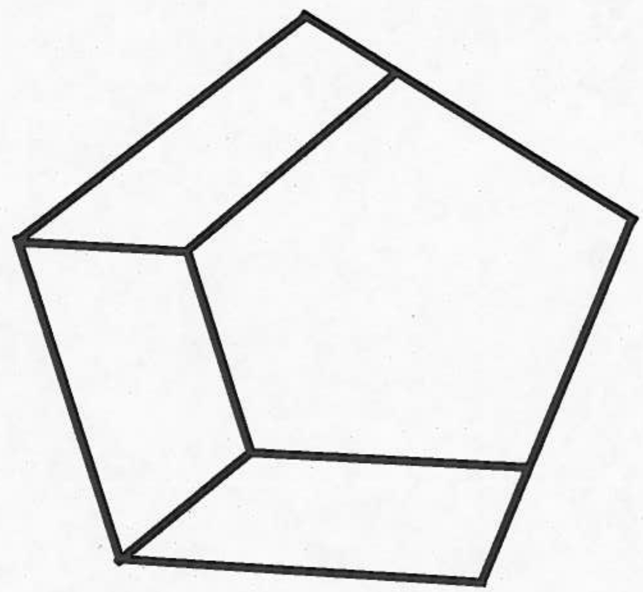

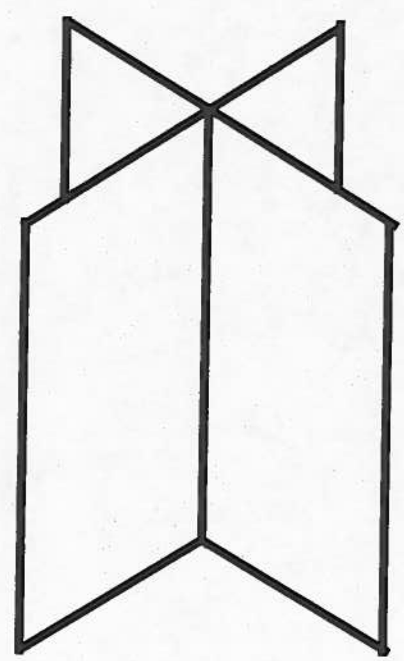

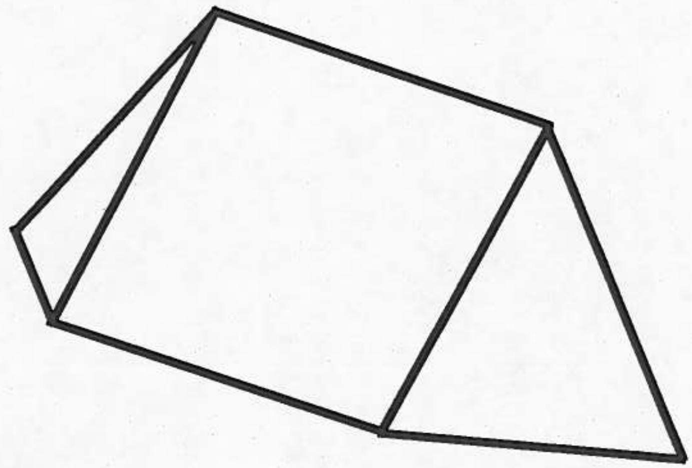

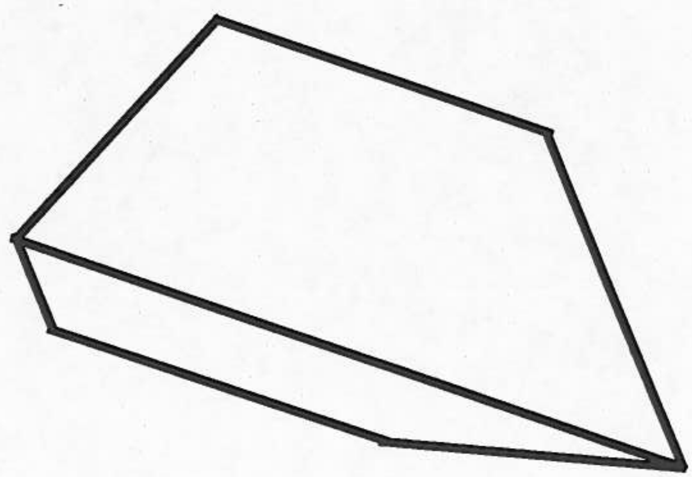

LEVEL 5

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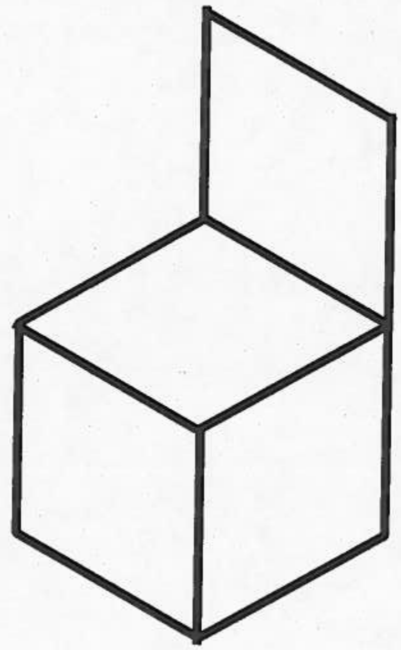
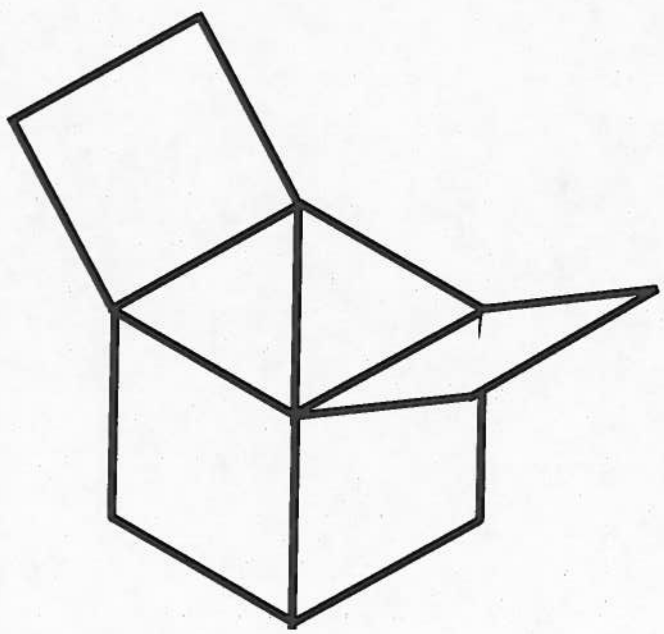
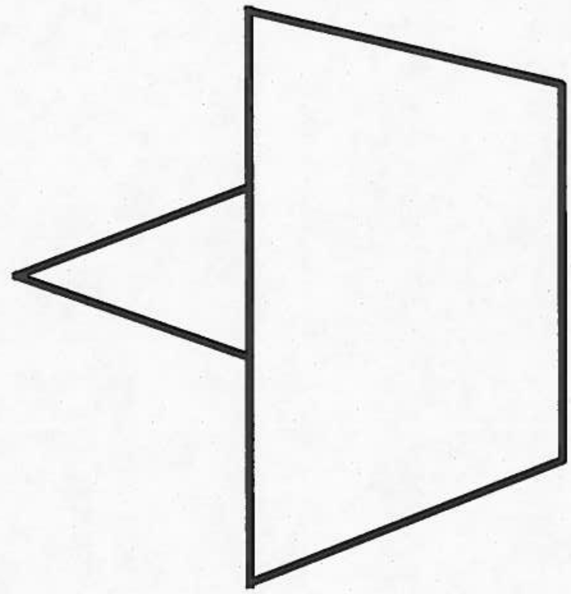
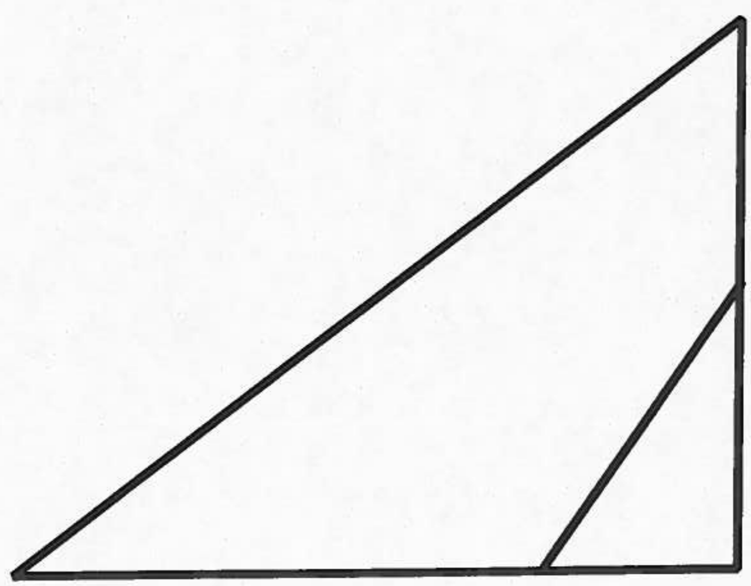
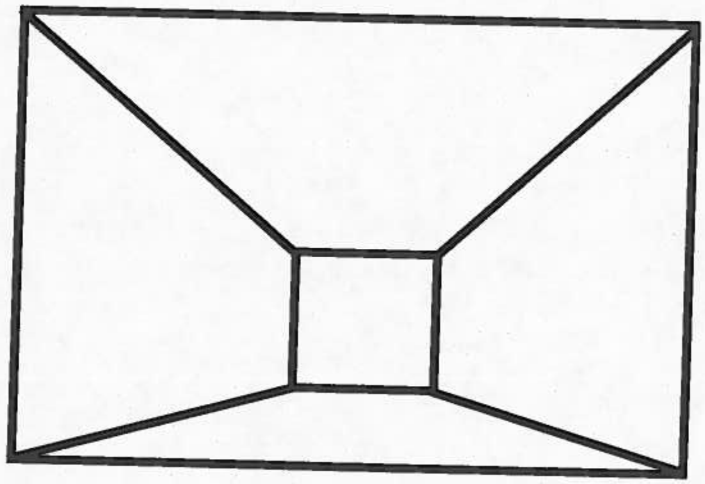
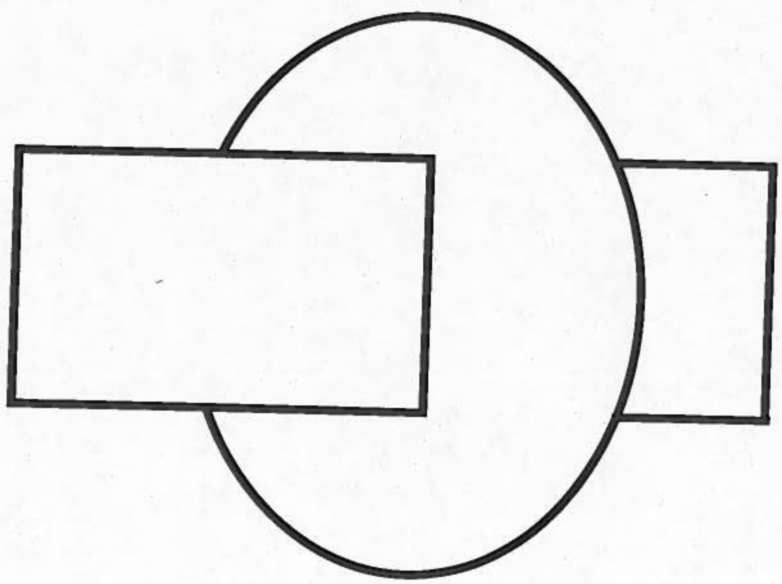


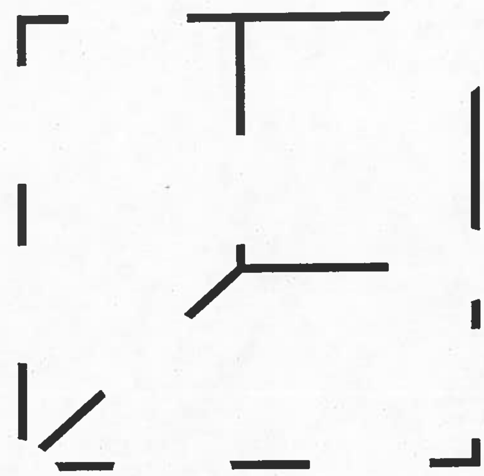


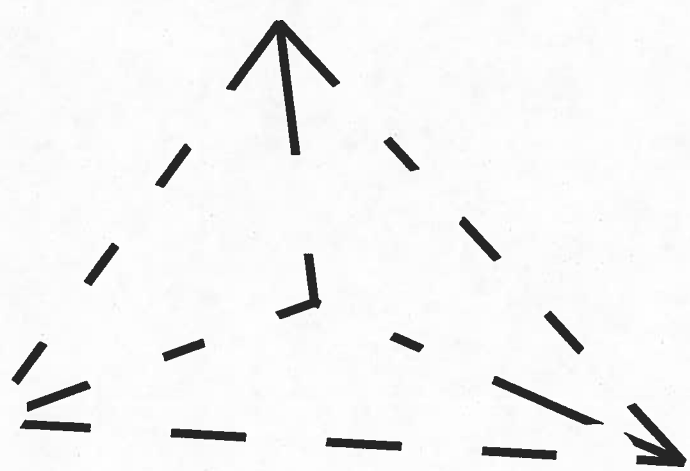


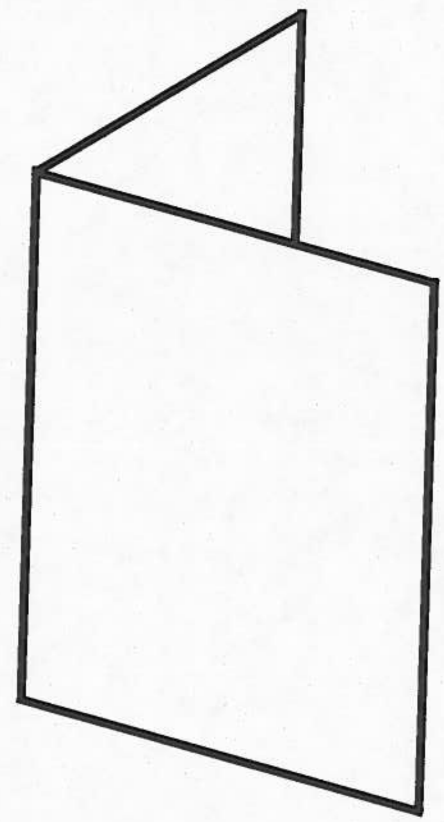


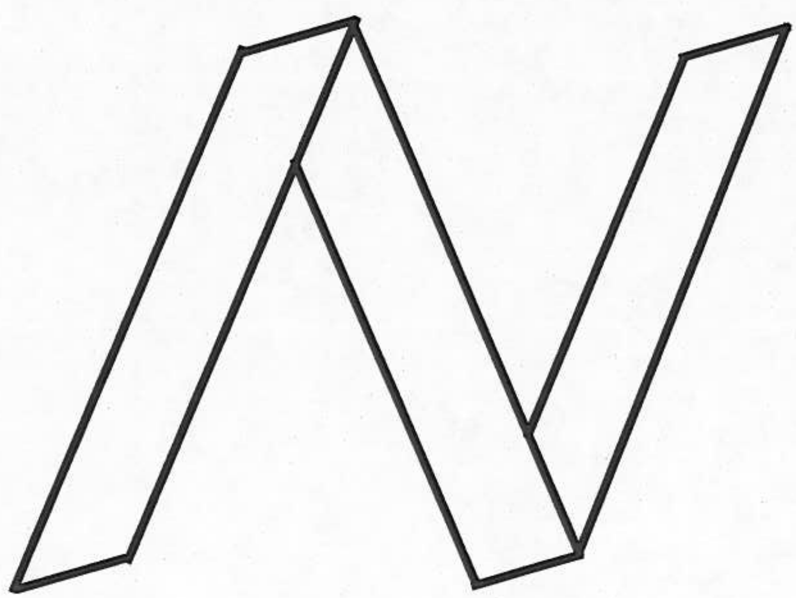


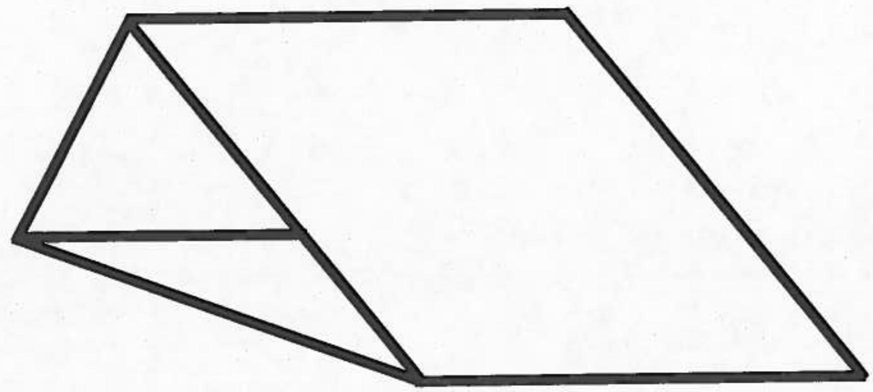


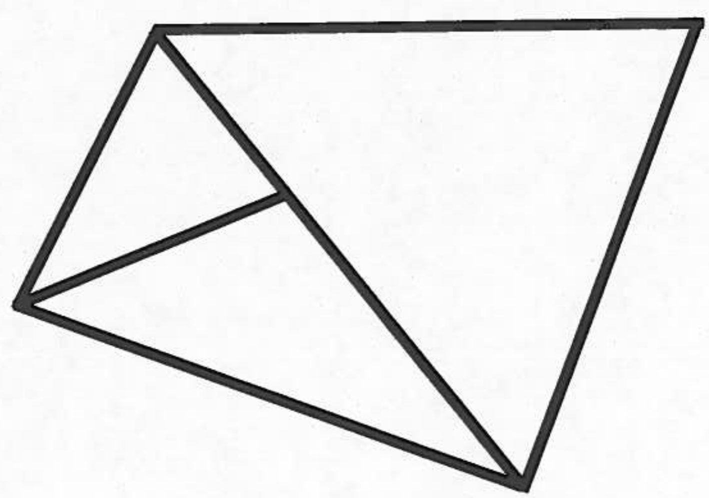


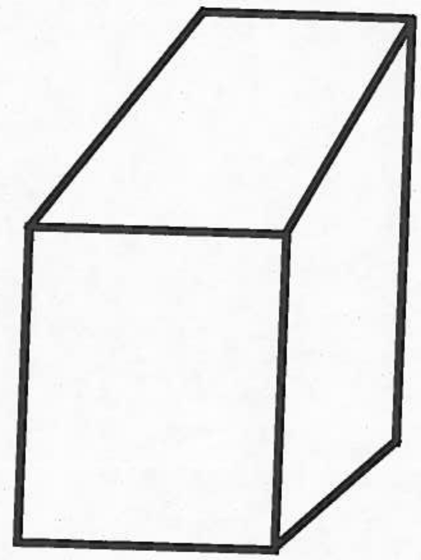


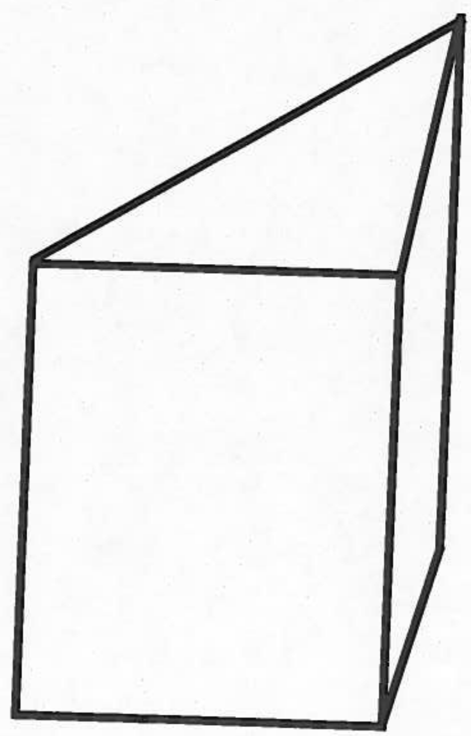


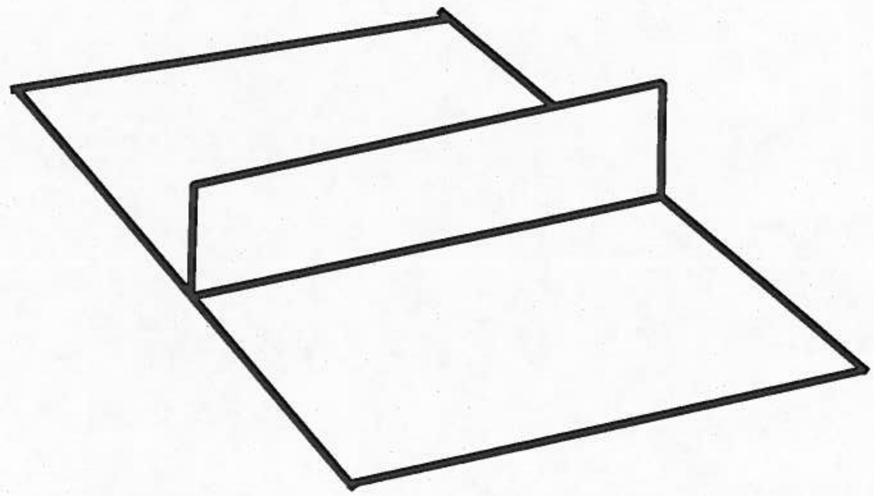


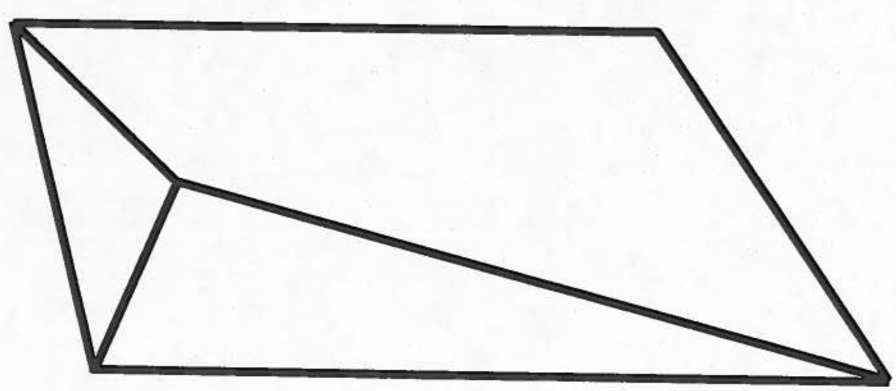


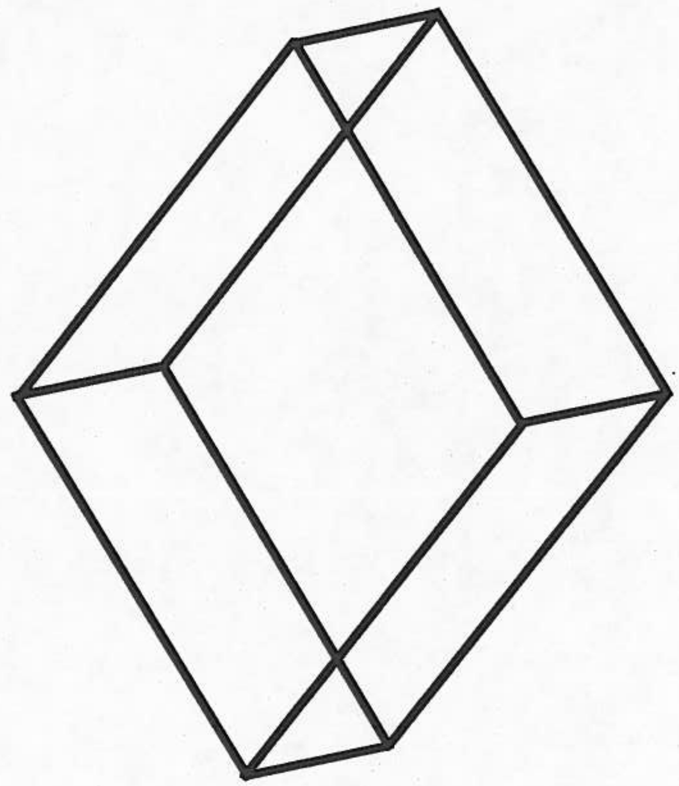


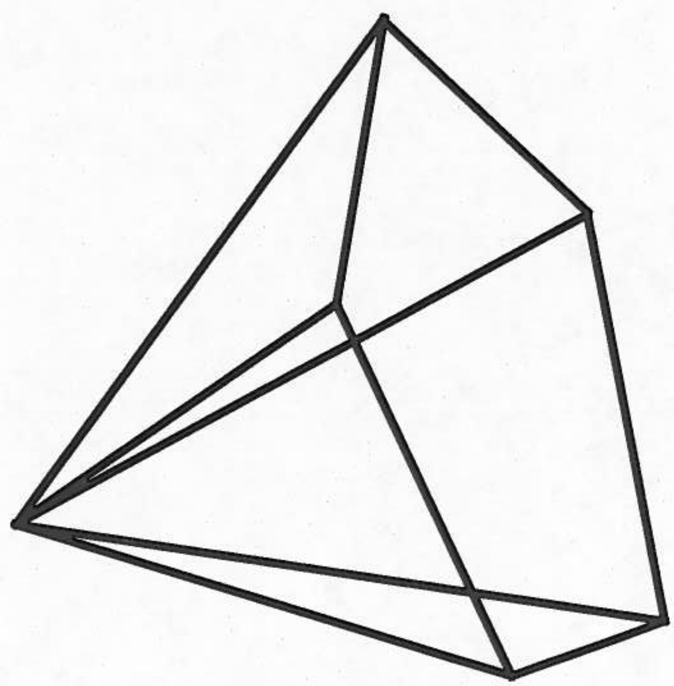


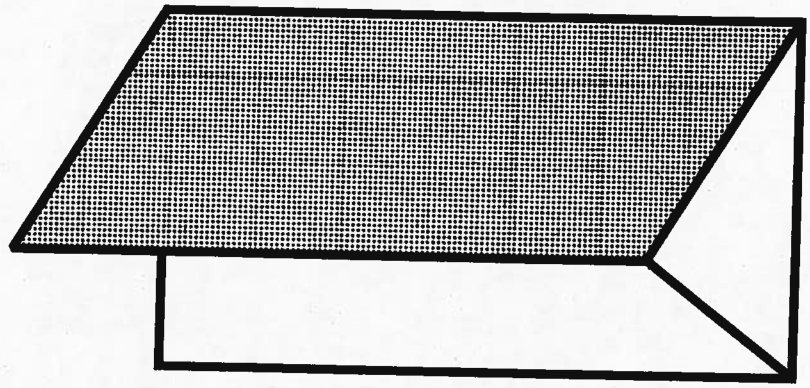


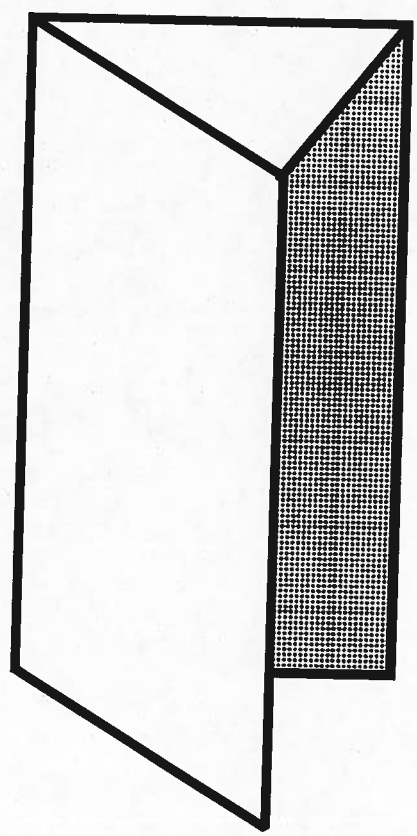


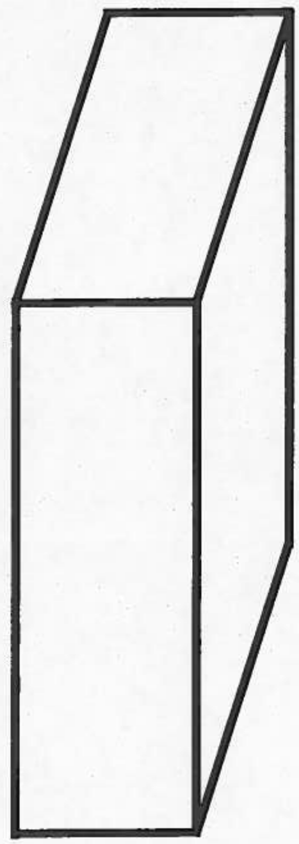


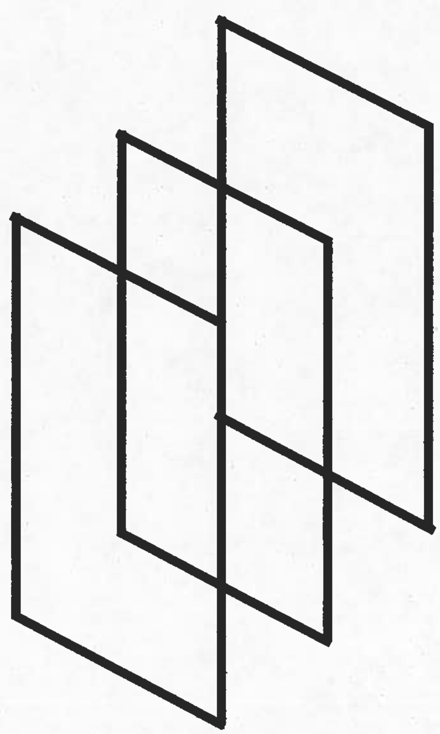






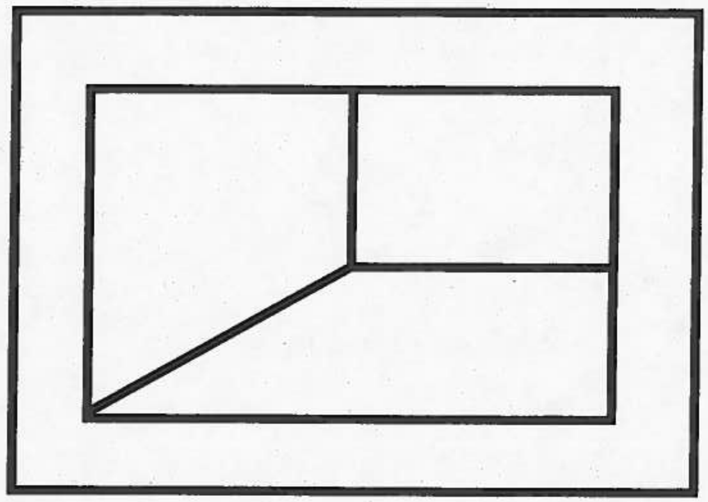


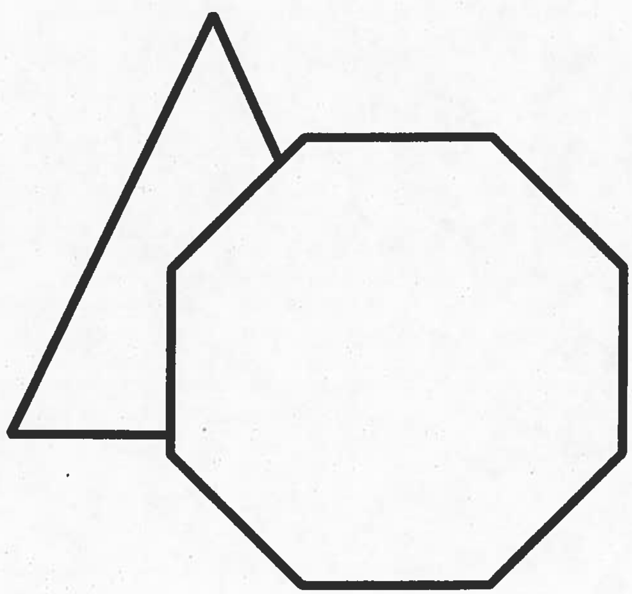


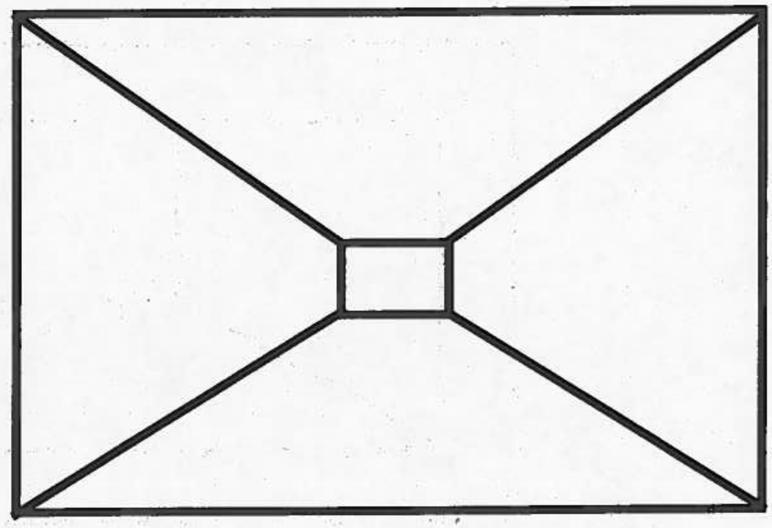


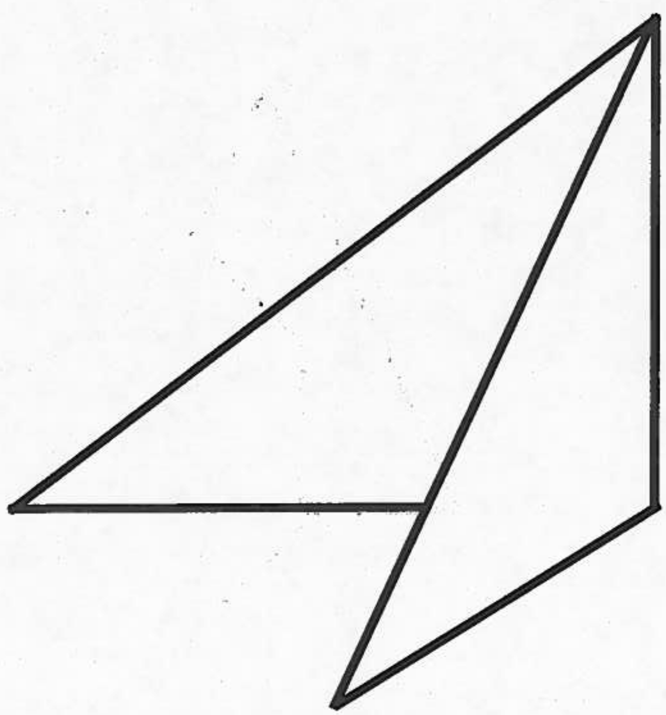
LEVEL 6

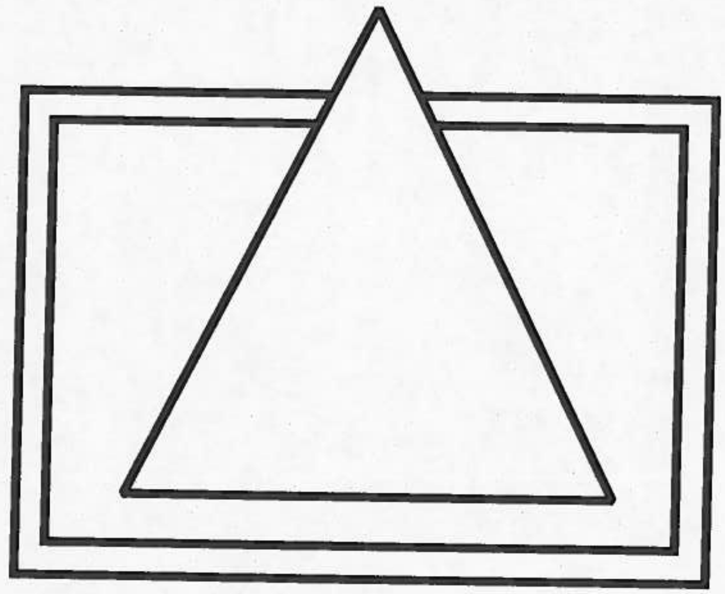
QUICK DRAW SHAPES

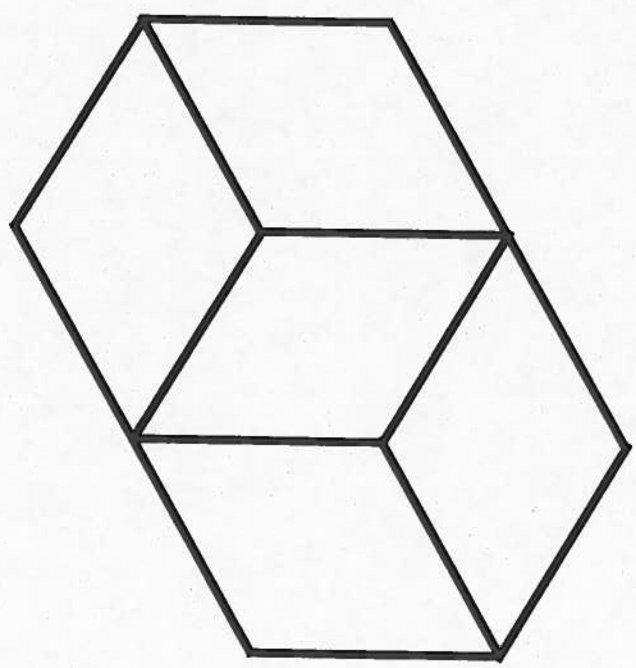


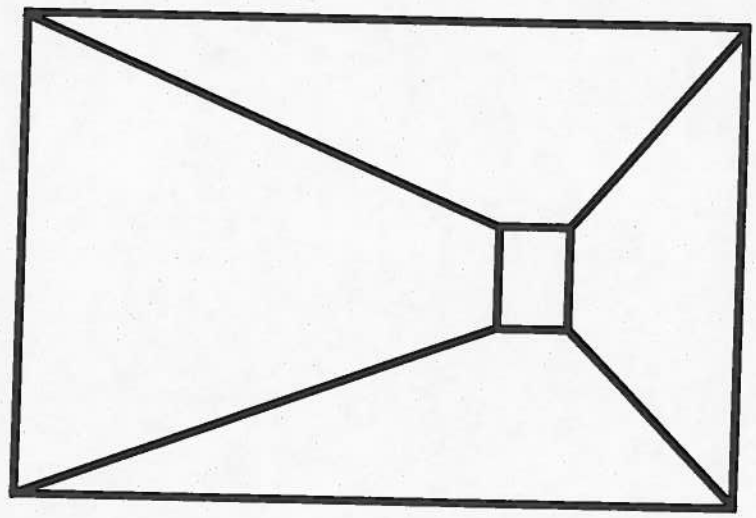


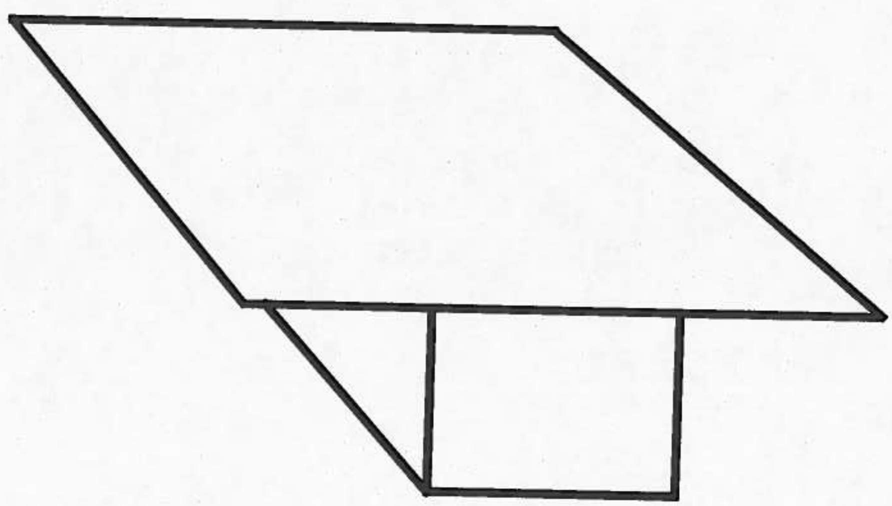


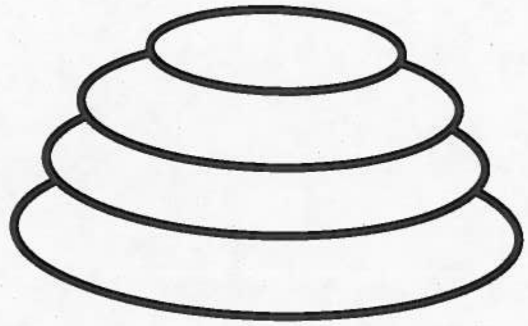


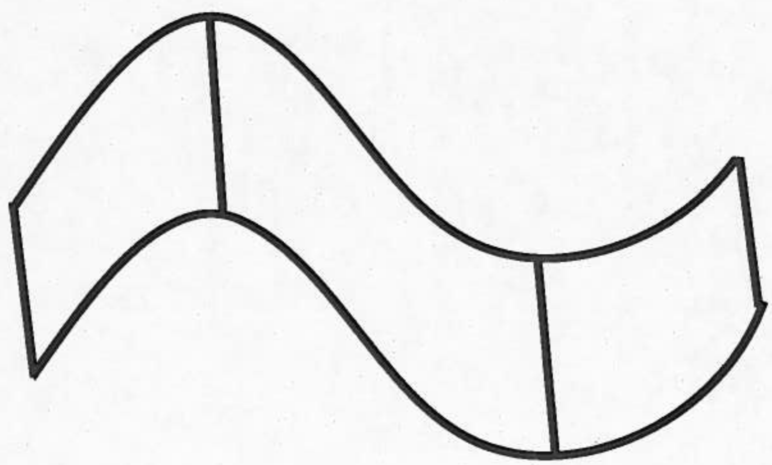


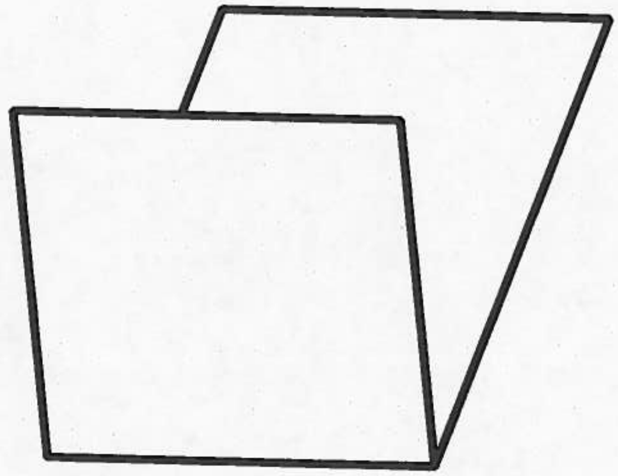


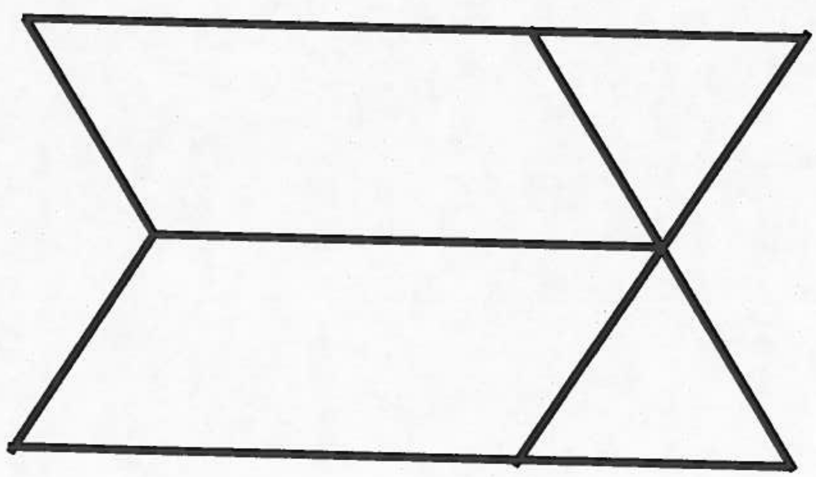


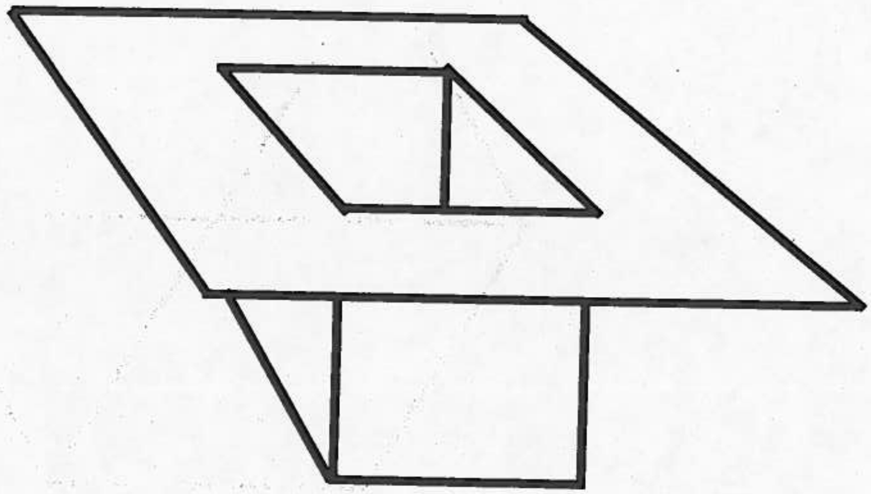


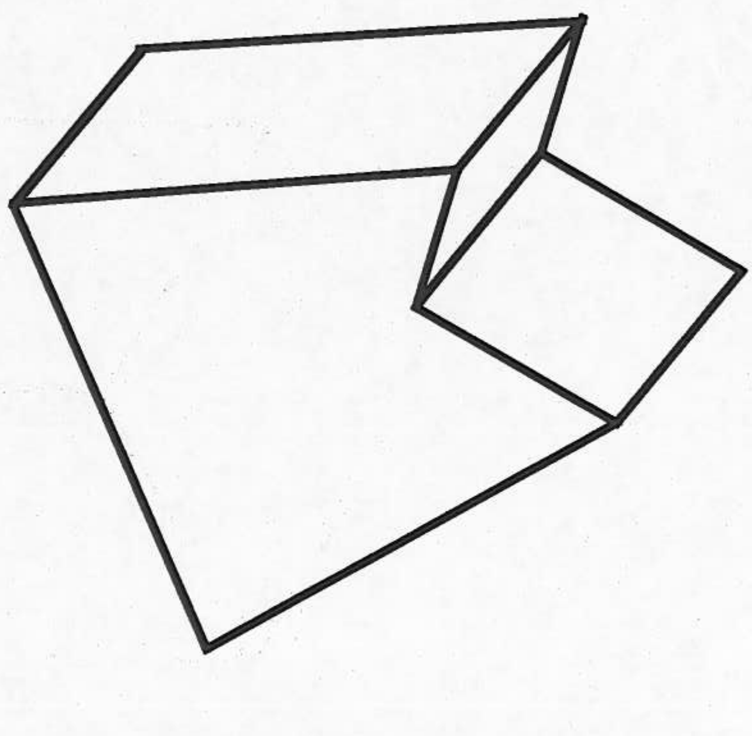


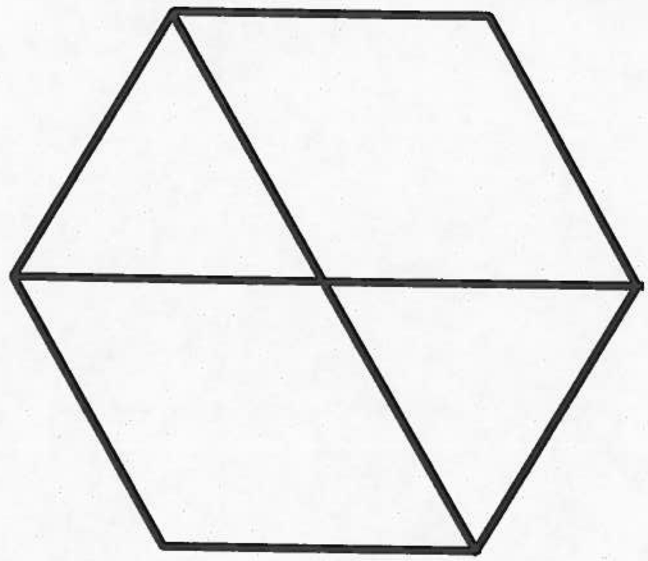


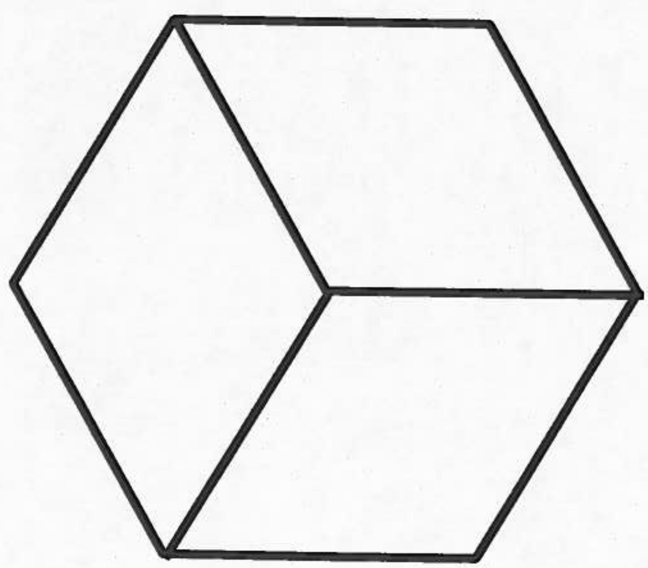


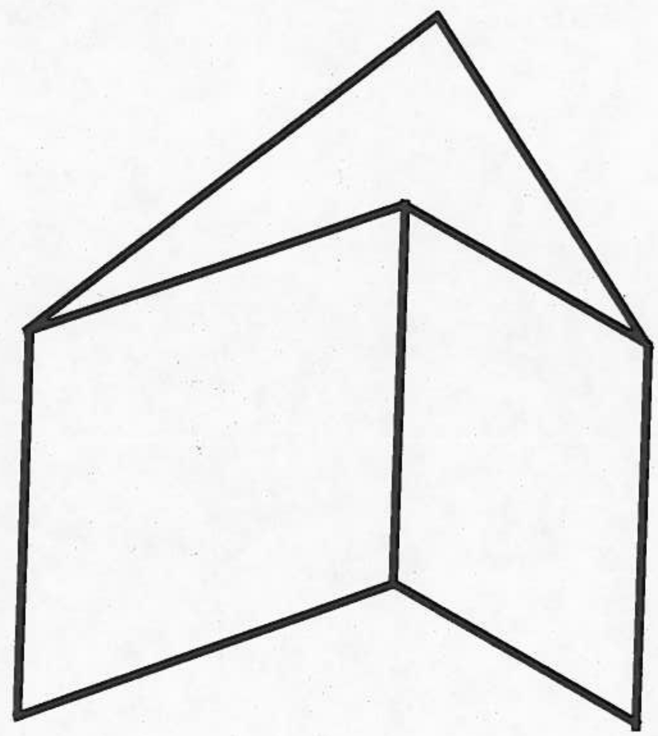


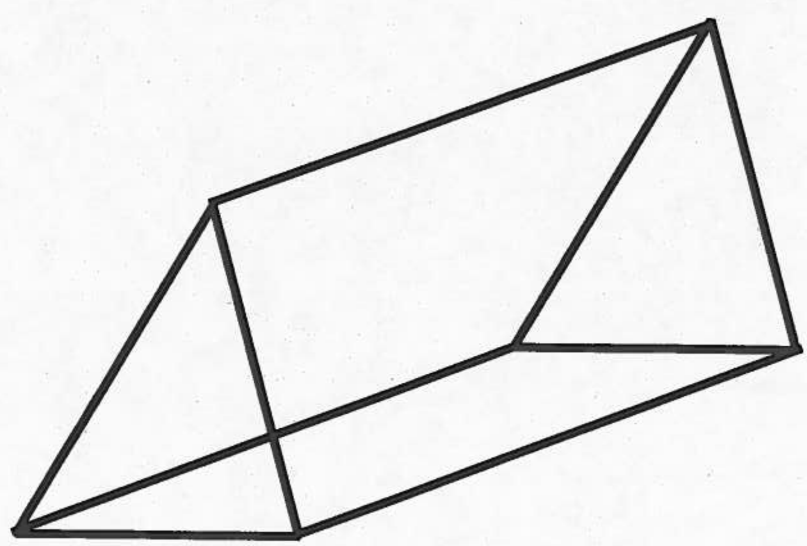


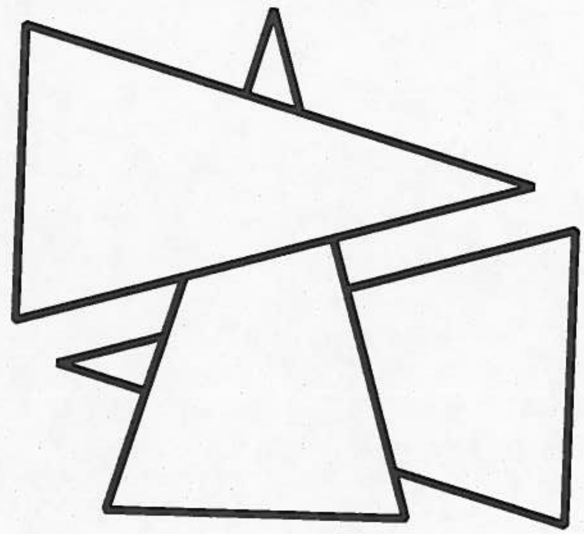


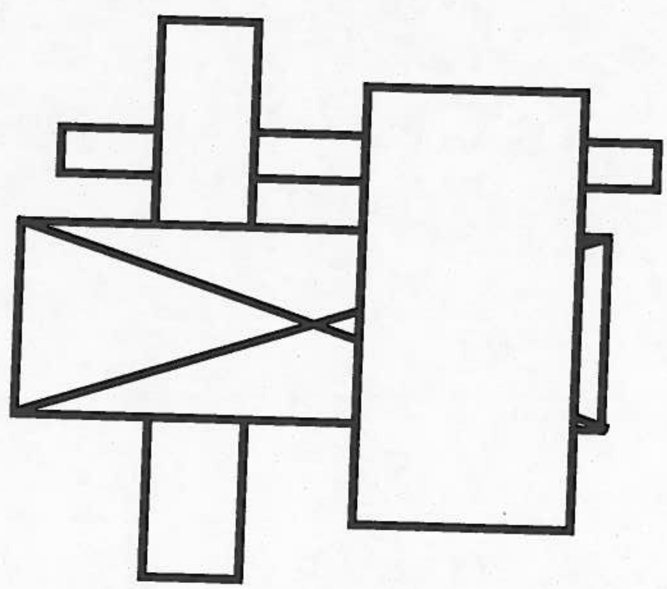


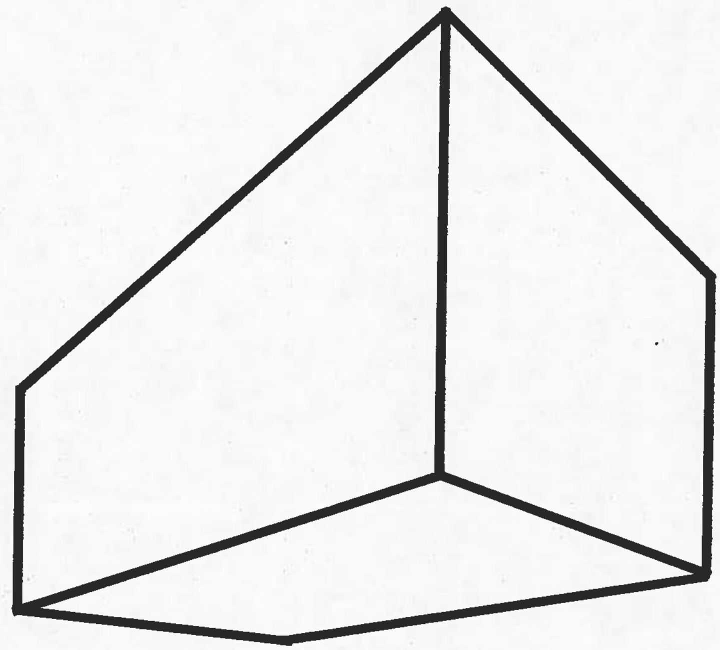


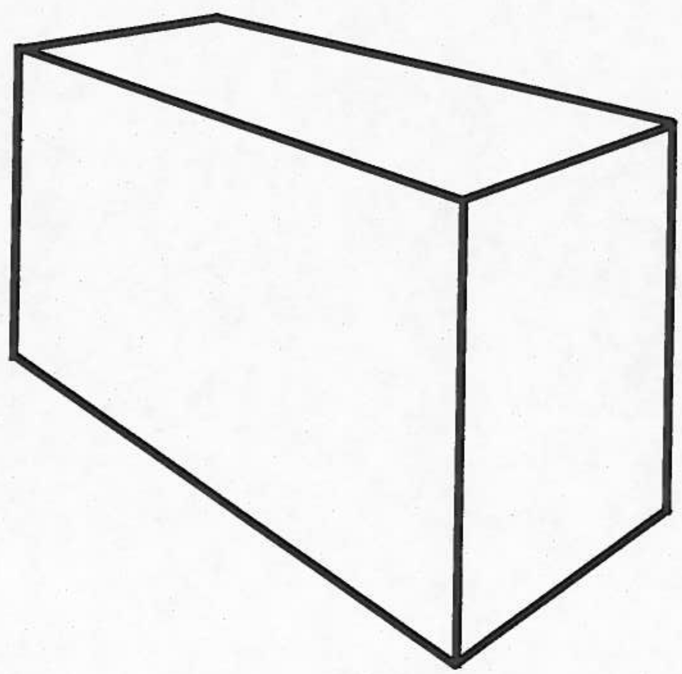


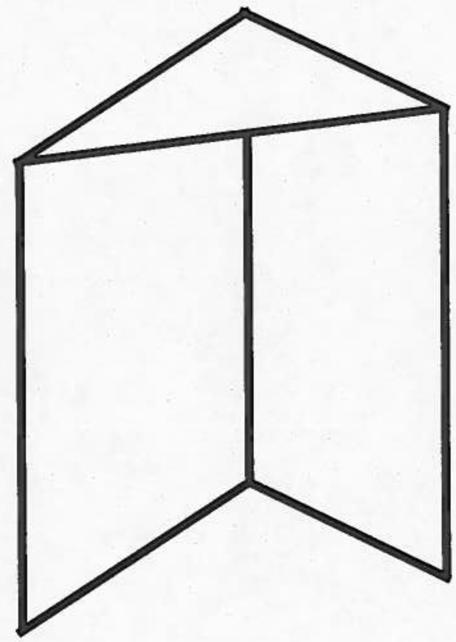


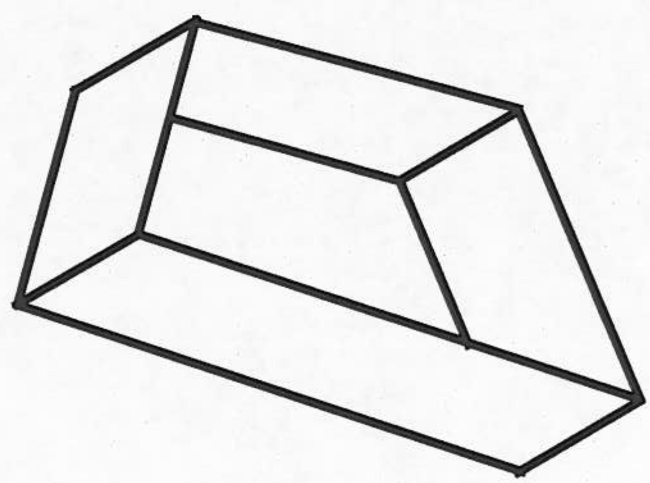












LEVEL 7

QUICK DRAW SHAPES